THE SECOND HOMOTOPY GROUP OF SPUN 2-SPHERES IN 4-SPACE

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1. **Introduction.** Andrews and Curtis [1] have shown that the second homotopy group of the complementary domain of a locally flat 2-sphere S^2 in the 4-sphere S^4 may not be trivial. This was shown to be the case if S^2 is formed by spinning the trefoil knot. Epstein [3] has shown that if S^2 is a spun nontrivial 2-sphere, then $\pi_2(S^4-S^2)$ is a free abelian group of infinite rank. Fox [6] has suggested that it might be more fruitful to consider the second homotopy group with its π_1-action, and has asked for an algorithm for calculating $\pi_2(S^4-S^2)$ as a $J\pi_1$-module. Sumners [8] has constructed a knotted 2-sphere in S^4 for which π_2 has nontrivial $J\pi_1$-torsion.

The following theorem gives the structure of π_2 as a $J\pi_1$-module for the case of spun 2-spheres.

THEOREM 2. *If $k(S^2)\subset S^4$ is a 2-sphere formed by spinning an arc A about the sphere S^2 and $(x_0, x_1, \cdots, x_n : r_1, r_2, \cdots, r_m)^\phi$ is a presentation of $\pi_1(S^4-k(S^2))$ with x_0 the image of the generator of $\pi_1(S^2-A)$ under the inclusion map, then*

$$\left(X_i(1 \leqq i \leqq n): \sum_{i=1}^{n} (\partial r_j/\partial x_i)*X_i = 0 \; (1 \leqq j \leqq m)\right)$$

is a presentation of $\pi_2(S^4-k(S^2))$ as a $J\pi_1$-module.

2. **Outline of proof.** Let S^n be the standard n-sphere. Let $S^n_\pm$ be the closed domains of S^n-S^{n-1}. Let A be an arc in S^3_+ which meets S^2 only in the end-points of A. Now rotate S^3_+ about S^2. Then A sweeps out a 2-sphere $k(S^2)$ called a spun 2-sphere [2].

THEOREM 1. *If $k(S^2)\subset S^4$ is a spun 2-sphere, then $\pi_2(S^4-k(S^2)) \simeq K/[K,K]$, where K is the kernel of the homomorphism $i_*: \pi_1(S^3-k(S^2)) \to \pi_1(S^4-k(S^2))$ induced by inclusion and $[K, K]$ is the commutator subgroup of K.*

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Proof in brief. Let $[f]\in K$, where $f\colon (S^1, 1)\to(S^3-kS^2, p)$. Since $[f]$ lies in the kernel of i_*, there exists a $g\colon (S^2, 1)\to(S^4-kS^2, p)$ such that

(1) $g|S^1=f$,
(2) $g(S^2_+)\subset S^4_+-kS^2$,
(3) $g(S^2_-)\subset S^4_- -kS^2$.

Define $\Phi\colon K\to\pi_2(S^4-k(S^2))$ as $\Phi[f]=[g]$. It follows from the asphericity of knots [7] that $\pi_2(S^4_\pm-k(S^2))=0$, and hence that $\Phi\colon K\to\pi_2(S^4-k(S^2))$ is a well-defined homomorphism. It can now be shown that Φ is onto and has $[K, K]$ as its kernel.

Note that the following sequences are exact:

$$1\to K\xrightarrow{j_*}\pi_1(S^3-k(S^2))\xrightarrow{i_*}\pi_1(S^4-k(S^2))\to 1$$
$$1\to[K,K]\to K\to\pi_2(S^4-k(S^2))\to 0.$$

Hence the action of $\pi_1(S^4-k(S^2))$ on $\pi_2(S^4-k(S^2))$ is obtained by lifting the elements of $\pi_1(S^4-k(S^2))$ by i_* to $\pi_1(S^3-k(S^2))$ and then applying the natural action of $\pi_1(S^3-k(S^2))$ on its normal subgroup K.

Let $(x_0, x_1, x_2, \cdots, x_n\colon r_1, r_2, \cdots, r_m)^\phi$ be a presentation of $\pi_1(S^4-k(S^2))$ with x_0 representing the image of the generator of $\pi_1(S^2-k(S^2))$ under the homomorphism $j_*\colon\pi_1(S^2-k(S^2))\to\pi_1(S^4-k(S^2))$ induced by inclusion. A corresponding presentation of $G=\pi_1(S^3-k(S^2))$ is $(x_0, x_{\pm1}, x_{\pm2}, \cdots, x_{\pm n}\colon r_{\pm1}, r_{\pm2}, \cdots, r_{\pm m})^\phi$, where $r_i(x_0, x_{-1}, \cdots, x_{-n})=r_{-i}$. Then K is the normal closure of $\{\phi(x_ix_{-i}^{-1})\}$ in G.

By means of the Reidemeister-Schreier theorem [5] it can be shown that:

LEMMA 7. *$(\{x_\alpha\}_{\alpha\in H}\colon\{r_\alpha\}_{\alpha\in H}, \{x_{-i\beta}(i\geqq 0)\}_{\beta\in H})$ is a presentation of K, where $H=\pi_1(S^4-k(S^2))$.*

Lifting the action of $\pi_1(S^4-k(S^2))$ on $\pi_2(S^4-k(S^2))$ up to this presentation, we have Theorem 1.

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CLASSIFICATION OF KNOTS IN CODIMENSION TWO

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Introduction. In this paper we consider smooth knots, i.e., smooth embeddings $\phi: S^n \to S^{n+2}$, $n \geqq 3$. Two knots ϕ and η are said to be equivalent if there is a diffeomorphism $f: S^{n+2} \to S^{n+2}$ such that $f\phi(S^n) = \eta(S^n)$. The embedding ϕ extends to an embedding $\bar{\phi}: S^n \times D^2 \to S^{n+2}$, and any two such extensions are ambient isotopic relative to $S^n \times 0$. Hence if $A = \mathrm{cl}(S^{n+2} - \bar{\phi}(S^n \times D^2))$, the pair $(A, \partial A)$ is determined up to diffeomorphism by the equivalence class of ϕ. We call $(A, \partial A)$ the complementary pair, or simply the complement, of the knot ϕ. In this paper we show that if $\pi_1 A$, the fundamental group of the knot, is infinite cyclic, then there is at most one knot inequivalent to ϕ with complementary pair $(B, \partial B)$ of the same homotopy type as $(A, \partial A)$. This result is of interest because for any $n \geqq 3$ there are many inequivalent knots $\phi: S^n \to S^{n+2}$ with fundamental group $\boldsymbol{Z}$, see for example [12]. (The result also holds in the P.L. case, provided ϕ extends to a P.L.-embedding $\bar{\phi}: S^n \times D^2 \to S^{n+2}$.)

1. Knots with diffeomorphic complements. In [4], Gluck showed that homeomorphisms of $S^2 \times S^1$ are isotopic if and only if they are homotopic and used this result to conclude that there are at most two knots $\phi: S^2 \to S^4$ with homeomorphic exteriors. In [1], W. Browder studied the pseudo-isotopy classes of diffeomorphisms (and P.L. equivalences) of $S^1 \times S^n$ for $n \geqq 5$. He showed that two P.L. equivalences are pseudo-isotopic if and only if they are homotopic. For the group $\mathfrak{D}(S^n \times S^1)$ of pseudo-isotopy classes of diffeomorphisms, he obtained the exact sequence