THE SECOND HOMOTOPY GROUP OF SPUN 2-SPHERES IN 4-SPACE

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Communicated by R. H. Bing, May 27, 1968

1. Introduction. Andrews and Curtis [1] have shown that the second homotopy group of the complementary domain of a locally flat 2-sphere \( S^2 \) in the 4-sphere \( S^4 \) may not be trivial. This was shown to be the case if \( S^2 \) is formed by spinning the trefoil knot. Epstein [3] has shown that if \( S^2 \) is a spun nontrivial 2-sphere, then \( \pi_2(S^4 - S^2) \) is a free abelian group of infinite rank. Fox [6] has suggested that it might be more fruitful to consider the second homotopy group with its \( \pi_1 \)-action, and has asked for an algorithm for calculating \( \pi_2(S^4 - S^2) \) as a \( J\pi_1 \)-module. Sumners [8] has constructed a knotted 2-sphere in \( S^4 \) for which \( \pi_2 \) has nontrivial \( J\pi_1 \)-torsion.

The following theorem gives the structure of \( \pi_2 \) as a \( J\pi_1 \)-module for the case of spun 2-spheres.

**Theorem 2.** If \( k(S^2) \subset S^4 \) is a 2-sphere formed by spinning an arc \( A \) about the sphere \( S^2 \) and \((x_0, x_1, \ldots, x_n: r_1, r_2, \ldots, r_m)^+ \) is a presentation of \( \pi_1(S^4 - k(S^2)) \) with \( x_0 \) the image of the generator of \( \pi_1(S^2 - A) \) under the inclusion map, then

\[
(X_i (1 \leq i \leq n): \sum_{i=1}^n (\partial r_j/\partial x_i) X_i = 0 \ (1 \leq j \leq m))
\]

is a presentation of \( \pi_2(S^4 - k(S^2)) \) as a \( J\pi_1 \)-module.

2. Outline of proof. Let \( S^n \) be the standard \( n \)-sphere. Let \( S^3_+ \) be the closed domains of \( S^n - S^{n-1} \). Let \( A \) be an arc in \( S^3_+ \) which meets \( S^2 \) only in the end-points of \( A \). Now rotate \( S^3_+ \) about \( S^2 \). Then \( A \) sweeps out a 2-sphere \( k(S^2) \) called a spun 2-sphere [2].

**Theorem 1.** If \( k(S^2) \subset S^4 \) is a spun 2-sphere, then \( \pi_2(S^4 - k(S^2)) \sim K/[K, K] \), where \( K \) is the kernel of the homomorphism \( i_*: \pi_1(S^3 - k(S^2)) \rightarrow \pi_1(S^4 - k(S^2)) \) induced by inclusion and \([K, K]\) is the commutator subgroup of \( K \).

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1 Supported by NSF Grant GP-5458.
Proof in brief. Let \([f] \in K\), where \(f: (S^1, 1) \to (S^3 - kS^3, p)\). Since \([f]\) lies in the kernel of \(i_*\), there exists a \(g: (S^2, 1) \to (S^4 - kS^2, p)\) such that

1. \(g|S^1 = f\),
2. \(g(S^2_+) \subseteq S^4 - kS^2\),
3. \(g(S^2_+) \subseteq S^4 - kS^2\).

Define \(\Phi: K \to \pi_3(S^4 - k(S^3))\) as \(\Phi[f] = [g]\). It follows from the asphericity of knots \([7]\) that \(\pi_2(S^3_+ - k(S^2)) = 0\), and hence that \(\Phi: K \to \pi_3(S^4 - k(S^3))\) is a well-defined homomorphism. It can now be shown that \(\Phi\) is onto and has \([K, K]\) as its kernel.

Note that the following sequences are exact:

\[
1 \to K \overset{j_*}{\to} \pi_1(S^3 - k(S^3)) \overset{i_*}{\to} \pi_1(S^4 - k(S^3)) \to 1
\]

\[
1 \to [K, K] \to K \to \pi_2(S^4 - k(S^3)) \to 0.
\]

Hence the action of \(\pi_1(S^4 - k(S^3))\) on \(\pi_2(S^4 - k(S^3))\) is obtained by lifting the elements of \(\pi_1(S^4 - k(S^3))\) by \(i_*\) to \(\pi_1(S^3 - k(S^2))\) and then applying the natural action of \(\pi_1(S^3 - k(S^2))\) on its normal subgroup \(K\).

Let \((x_0, x_1, x_2, \ldots, x_n, r_1, r_2, \ldots, r_m)^*\) be a presentation of \(\pi_1(S^4 - k(S^3))\) with \(x_0\) representing the image of the generator of \(\pi_1(S^2 - k(S^2))\) under the homomorphism \(j_*: \pi_1(S^3 - k(S^3)) \to \pi_1(S^4 - k(S^3))\) induced by inclusion. A corresponding presentation of \(G = \pi_1(S^3 - k(S^2))\) is \((x_0, x_1, x_2, \ldots, x_n, r_1, r_2, \ldots, r_m)^*\), where \(r_i(x_0, x_1, \ldots, x_n) = r_{-i}\). Then \(K\) is the normal closure of \(\{\phi(x_{\pm i})\}\) in \(G\).

By means of the Reidemeister-Schreier theorem \([5]\) it can be shown that:

**Lemma 7.** \((\{x_0\}_{\alpha \in H}, \{r_{\alpha}\}_{\alpha \in H}, \{x_{-i} (i \geq 0)\}_{\beta \in H})\) is a presentation of \(K\), where \(H = \pi_1(S^3 - k(S^2))\).

Lifting the action of \(\pi_1(S^4 - k(S^3))\) on \(\pi_2(S^4 - k(S^3))\) up to this presentation, we have Theorem 1.

**References**

CLASSIFICATION OF KNOTS IN CODIMENSION TWO

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Communicated by W. Browder, May 31, 1968

Introduction. In this paper we consider smooth knots, i.e., smooth embeddings \( \phi: S^n \rightarrow S^{n+2}, \ n \geq 3 \). Two knots \( \phi \) and \( \eta \) are said to be equivalent if there is a diffeomorphism \( f: S^{n+2} \rightarrow S^{n+2} \) such that \( f_\* (S^n) = \eta (S^n) \). The embedding \( \phi \) extends to an embedding \( \tilde{\phi}: S^n \times D^2 \rightarrow S^{n+2} \), and any two such extensions are ambient isotopic relative to \( S^n \times 0 \). Hence if \( A = \text{cl}(S^{n+2} - \tilde{\phi}(S^n \times D^2)) \), the pair \( (A, \partial A) \) is determined up to diffeomorphism by the equivalence class of \( \phi \). We call \( (A, \partial A) \) the complementary pair, or simply the complement, of the knot \( \phi \). In this paper we show that if \( \pi_1 A \), the fundamental group of the knot, is infinite cyclic, then there is at most one knot inequivalent to \( \phi \) with complementary pair \( (B, \partial B) \) of the same homotopy type as \( (A, \partial A) \). This result is of interest because for any \( n \geq 3 \) there are many inequivalent knots \( \phi: S^n \rightarrow S^{n+2} \) with fundamental group \( \mathbb{Z} \), see for example [12]. (The result also holds in the P.L. case, provided \( \phi \) extends to a P.L.-embedding \( \tilde{\phi}: S^n \times D^2 \rightarrow S^{n+2} \).)

1. Knots with diffeomorphic complements. In [4], Gluck showed that homeomorphisms of \( S^2 \times S^4 \) are isotopic if and only if they are homotopic and used this result to conclude that there are at most two knots \( \phi: S^2 \rightarrow S^4 \) with homeomorphic exteriors. In [1], W. Browder studied the pseudo-isotopy classes of diffeomorphisms (and P.L. equivalences) of \( S^1 \times S^n \) for \( n \geq 5 \). He showed that two P.L. equivalences are pseudo-isotopic if and only if they are homotopic. For the group \( \mathcal{D}(S^1 \times S^n) \) of pseudo-isotopy classes of diffeomorphisms, he obtained the exact sequence