DUALITY AND ASYMPTOTIC SOLVABILITY OVER CONES[1]

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Introduction. A duality theory for linear programming over closed convex cones is developed by using the solvability theorem of [1]. A complete classification of all duality states is given, which includes Duffin's duality theorem (1 of [3]), together with asymptotic refinements. For convenience, our results are stated in R^n, but extensions to more general spaces considered in [1] are possible. Complete proofs of all results will be given elsewhere. The main ingredients of the duality theory over closed convex cones are essentially due to Duffin as set forth in his fundamental paper [3]; see also Kretschmer [5].

0. Notations and preliminaries. C—a closed convex cone in R^n,

$$C^* = \{y \in R^n : x \in C \Rightarrow (y, x) \geq 0\}.$$

For the system

(1) $$Ax = b \qquad x \in C$$

with given $A \in R^{m \times n}$, $b \in R^m$ let

$$F(1) = \{x \in C : Ax = b\},$$

$$A F(1) = \{ \{x_k : k = 1, 2, \cdots \} : x_k \in C, \lim_k Ax_k = b\}.$$

$A F(1)$ is a set of sequences. If $F(1) \neq \varnothing$ then it can be imbedded naturally in $A F(1)$.

DEFINITION. (1) is

CONS (consistent) if $F(1) \neq \varnothing$,

INC (inconsistent) if $F(1) = \varnothing$,

AC (asymptot. consist.) if $A F(1) \neq \varnothing$,

SINC (strongly inconsist.) if $A F(1) = \varnothing$.

We use the following

SOLVABILITY THEOREM ([1, p. 543]). *The following are equivalent*:
(i) $Ax = b$, $x \in C$ is AC,
(ii) $A^t y \in C^* \Rightarrow (b, y) \geq 0$.

Consider now the pair of problems
   (I.C) $\sup(c, x)$ subject to $Ax = b$, $x \in C$,
   (II.C*) $\inf(b, y)$ subject to $A^t y - c \in C^*$
with given $A \in R^{m \times n}$, $b \in R^m$, $c \in R^n$ and the associated systems

(1) $$Ax = b, \qquad x \in C,$$

$$(A^t, -I)\begin{pmatrix} y \\ z \end{pmatrix} = c,$$

(2)

$$\begin{pmatrix} y \\ z \end{pmatrix} \in R^m \times C^*.$$

DEFINITION.[2] (i) (I.C)[(II.C*)] is CONS, INC, AC or SINC if so is (1)[(2)].

(ii) Let (1)[(2)] be AC. Then (I.C)[(II.C*)] is PAC (properly AC) if $\exists \{x_k\} \in AF(1) \ni \lim_k \sup (c, x_k) > -\infty$ [$\{(y_k, z_k)\} \in AF(2) \ni \lim_k \inf(b, y_k) < \infty$]; otherwise it is IAC (improperly AC).

(iii) Let (I.C)[(II.C*)] be CONS. Then it is BD (bounded) if $\sup\{(c, x): x \in F(1)\} < \infty$ [$\inf\{(b, y): (y, z) \in F(2)\} > -\infty$]. Otherwise it is UBD (unbounded).

(iv) Let (I.C)[(II.C*)] be PAC. Then it is ABD (asymptotically BD) if $\sup\{\lim_k \sup(c, x_k): \{x_k\} \in AF(1)\} < \infty$

$$[\inf \{\lim_k \inf (b, y_k): \{y_k, z_k\} \in AF(2)\} > -\infty].$$

Otherwise it is AUBD (asymptotically UBD).

1. The duality states. Of the 49 mutually exclusive and collectively exhaustive states for problems (I.C) and (II.C*) only 10 are possible, and are those denoted in the table below by positive integers. A zero in the table means that the corresponding state is impossible. This classification constitutes our theorem, whose proof is given in two parts. Part 1 proves the possible states by examples so numbered that the possibility of each state is demonstrated by the example bearing the same number. Part 2 excludes the impossible states by a series of lemmas.

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[2] In order to refine and extend the original cases developed by Duffin [3], we take the liberty of deviating from his terminology.

THEOREM.

(II. C*) \ (I. C)	AC · PAC · CONS · ABD · BD	AC · PAC · CONS · AUBD · UBD	AC · PAC · INC · ABD	AC · PAC · INC · AUBD	AC · PAC · INC · IAC	AC · PAC · INC · SINC	SINC
AC · PAC · CONS · ABD · BD	1	0	0	0	0	0	0
AC · PAC · CONS · AUBD · UBD	0	0	0	2	0	0	0
(AC · PAC · CONS · UBD)	0	0	0	0	0	3	4
AC · PAC · INC · ABD	0	5	0	0	0	0	0
AC · PAC · INC · AUBD	0	0	0	0	0	0	6
AC · IAC · IAC	0	0	7	0	0	0	0
SINC · SINC	0	0	8	0	9	0	10

PROOF. *Part 1. The possible states:*

Let C be the cone in R^3 of all vectors forming an angle $\leq 45°$ with the vector $(1, 0, 1)$. Then $C = C^*$. In all 10 examples below we use this C and one of the following matrices

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad A_2 = (0 \quad 0 \quad 1)$$

for A.

Example 1. $A = A_1$, $b = (0, 1)$, $c = (0, 0, 1)$,

Example 2. $A = A_1$, $b = (1, 0)$, $c = (0, 0, 0)$,

Example 3. $A = A_1$, $b = (1, 0)$, $c = (-1, 0, 0)$,

Example 4. $A = A_2$, $b = (-1)$, $c = (-1, 0, 0)$,

Example 5. $A = A_2$, $b = (0)$, $c = (0, -1, 0)$,

Example 6. $A = A_2$, $b = (-1)$, $c = (0, 1, 0)$,

Example 7. $A = A_2$, $b = (1)$, $c = (0, 1, 0)$,

Example 8. $A = A_2$, $b = (1)$, $c = (1, 0, 0)$,

Example 9. $A = A_1$, $b = (1, 0)$, $c = (1, 0, 0)$,
Example 10. $A = A_2$, $b = (-1)$, $c = (1, 0, 0)$.
Part 2. *The impossible states*:

LEMMA 1. (I.C) AC *and* (II.C*) $CONS \Rightarrow$

(3) $\sup \{\lim \sup_k (c, x_k) : \{x_k\} \in AF(1)\} \leqq \inf \{(b, y) : (y, z) \in F(2)\}$.

Moreover if (II.C*) *is also BD then* $\exists$ *a sequence* $\{x_k\} \in AF(1) \ni$

(4) $\lim \sup_k (c, x_k) = \inf \{(b, y) : (y, z) \in F(2)\}$.

PROOF. To prove (3) let $\{x_k\} \in AF(1)$ and $A^t y - c = z \in C^*$. Then

$$(b, y) = \lim_k (Ax_k, y) = \lim_k (x_k, A^t y) = \lim_k (x_k, c + z)$$

$$= \lim \sup_k (x_k, c + z)$$

$$\geq \lim \sup_k (x_k, c) \quad \text{since } x_k \in C, z \in C^*,$$

which proves (3).

To prove the existence of a maximizing sequence $\{x_k\} \in AF(1)$ let

(5) $\beta = \inf \{(b, y) : (y, z) \in F(2)\}$.

Since (II.C*) is BD it follows that

(6)
$$\begin{pmatrix} A^t & -I & -c \\ 0 & I & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y \\ z \\ z_{n+1} \end{pmatrix} \in \{0\} \times C^* \times R_+$$

$$\Rightarrow \left(\left(\begin{pmatrix} b \\ 0 \\ -\beta \end{pmatrix}, \begin{pmatrix} y \\ z \\ z_{n+1} \end{pmatrix} \right) \right) \geqq 0.$$

For if $z_{n+1} > 0$ then by setting $z_{n+1} = 1$ (6) follows from the definition
(5) of β. If on the other hand $z_{n+1} = 0$ and (6) is false, i.e. $\exists (y^0, z^0) \ni$

$$\begin{pmatrix} A^t & -I & -c \\ 0 & I & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y^0 \\ z^0 \\ 0 \end{pmatrix} \in \{0\} \times C^* \times R_+, \quad \left(\left(\begin{pmatrix} b \\ 0 \\ -\beta \end{pmatrix}, \begin{pmatrix} y^0 \\ z^0 \\ 0 \end{pmatrix} \right) \right) < 0,$$

then for any $(y, z) \in F(2)$ we take $(y_k, z_k) = (y, z) + k(y^0, z^0)$, $k = 1$,
$2, \cdots$ and verify that

$$(y_k, z_k) \in F(2) \qquad \forall k.$$

But $(b, \, y_k) \to -\infty$, which contradicts (II.C*) BD. Finally (6) is equivalent to

$$
\begin{pmatrix} A & 0 & 0 \\ -I & I & 0 \\ -c^t & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ u \\ v \end{pmatrix} = \begin{pmatrix} b \\ 0 \\ -\beta \end{pmatrix}, \qquad \begin{pmatrix} x \\ u \\ v \end{pmatrix} \in R^n \times C \times R_+
$$

being AC, which proves the existence of $\{x_k\} \in AF(2)$ satisfying (4). $\square$

LEMMA 2. (I.C) *CONS and* (II.C*)

$$
AC \Rightarrow \sup \{(c, x) : x \in F(1)\} \le \inf \{\liminf_k (b, y_k) : \{y_k, z_k\} \in AF(2)\}.
$$

Moreover if (I.C) *is also BD then* $\exists$ *a sequence*

$$
\{y_k, z_k\} \in AF(2) \ni \sup \{(c, x) : x \in F(1)\} = \liminf_k (b, y_k).
$$

LEMMA 3. (I.C) *PAC and ABD*$\Rightarrow$(II.C*) *CONS and BD.*

LEMMA 4. (II.C*) *PAC and ABD*$\Rightarrow$(I.C) *CONS and BD.*

LEMMA 5. *One of the problems* (I.C) *and* (II.C*) *is CONS and ABD iff so is the other, in which case*

(7)
$$
\sup \{\limsup_k (c, x_k) : \{x_k\} \in AF(1)\} = \inf \{(b, y) : (y, z) \in F(2)\}
$$

$$
\ge \inf \{\liminf_k (b, y_k) : \{y_k, z_k\} \in AF(2)\} = \sup \{(c, x) : x \in F(1)\}.
$$

LEMMA 6. (I.C) *IAC*$\Rightarrow$(II.C*) *CONS and UBD.*

PROOF. We prove first that (I.C) IAC$\Rightarrow$(II.C*) CONS. Assume to the contrary that (II.C*) is INC. Then for any $r > 0$

$$
(A^t, \, -c) \begin{pmatrix} y \\ y_{m+1} \end{pmatrix} \in C^* \Rightarrow \left(\begin{pmatrix} y \\ y_{m+1} \end{pmatrix}, \begin{pmatrix} 0 \\ -r \end{pmatrix} \right) \ge 0.
$$

By the solvability theorem this is equivalent to

(8.r)
$$
\begin{pmatrix} A \\ -c^t \end{pmatrix} u = \begin{pmatrix} 0 \\ -r \end{pmatrix}, \qquad u \in C
$$

being AC.

Let now $\{x_k\} \in AF(1)$ and

$$
r_k = |(c, x_k)| \qquad k = 1, 2, \cdots.
$$

For each k let $\{u_{kj}\} \in AF(8.r_k)$ of which we choose an element

$$u_{kj(k)}$$

satisfying

$$\left| \frac{r_k A u_{kj(k)}}{(c, u_{kj(k)})} \right| < \frac{1}{2^k}, \qquad k = 1, 2, \cdots .$$

The sequence

$$\{v_k\} = x_k + r_k \frac{u_{kj(k)}}{(c, u_{kj(k)})}, \qquad k = 1, 2, \cdots ,$$

satisfies

$$\{v_k\} \in AF(1)$$

but

$$(c, v_k) = (c, x_k) + |(c, x_k)| \geqq 0$$

contradicting (I.C) IAC. Finally (II.C*) is UBD. For if it is BD then by Lemma 1 (I.C) cannot be IAC.   □

LEMMA 7. (II.C*) $IAC \Rightarrow$ (I.C) CONS and UBD.

The zero-numbered states in the above table are excluded by applying Lemmas 1–7. This completes the proof of the theorem.   □

2. **Remarks.** (i) If the cone C (hence C^*) is polyhedral then $F(1) = AF(1)$ and $F(2) = AF(2)$ so that both problems (I.C) and (II.C*) are either CONS or SINC. Only the 4 states, nos. 1, 4, 8 and 10, are then possible. For C polyhedral our results thus give (an alternate proof of) the duality theorem of linear programming, e.g. [4], [2].

(ii) The following simple example shows the possibility of arbitrary nonnegative values for the difference between the sides of inequality (7) in Lemma 5.

Let $C = C^*$ be the cone given in Part 1 of the proof and let

$$\bar{C} = C \times R_+ = C^*.$$

For any $\alpha \leqq 0$, let the problem (I.$\tilde{C}$) be given as

$$\text{maximize} - x_2$$

$$\text{s.t.} \qquad x_3 \qquad = 0$$

$$x_2 \qquad - x_4 = \alpha$$

$$(x_1, x_2, x_3, x_4) \in \tilde{C}.$$

The problem (II.$\tilde{C}$*) is

$$\text{minimize } \alpha y_2$$
$$\text{s.t.} \qquad (0, 1 + y_2, y_1, - y_2) \in \tilde{C}.$$

Both problems are CONS and ABD. Furthermore

$$\sup \left\{ \limsup (c, x_k) : \{x_k\} \in AF(1) \right\} = \inf \left\{ (b, y) : (y, z) \in F(2) \right\} = -\alpha$$

and

$$\sup \left\{ (c, x) : x \in F(1) \right\} = \inf \left\{ \liminf_k \left\{ (b, y_k) : \{y_k, z_k\} \in AF(2) \right\} \right\} = 0.$$

The difference in (7) is thus $-\alpha$.

(iii) Lemmas 1, 2 and 5 reprove Duffin's duality theorem (1 of [3]).

(iv) The above results show that if both problems (I.C) and (II.C*) are INC, then at least one of them is SINC. It is thus impossible for both problems to be AC but INC.

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