BAXTER ALGEBRAS AND COMBINATORIAL IDENTITIES. II

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1. **Introduction.** The problem of obtaining an analog of the Baxter-Bohnenblust-Spitzer formula for cyclic permutations was suggested to me by Mark Kac. Its solution is presented here; the method leads in fact to similar identities for any group of permutations; the main result is identity (7) of §5. The tools of the proof are the result of I, which reduces computations with Baxter operators to computations with symmetric functions, and Möbius inversion on the lattice of periods of a group action (definition below).

To be sure, the present results are more of combinatorial than of probabilistic interest; further applications, with special regards to the Baxter algebras arising in probability, will be given in the third part.

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2. **Notation.** A partition π of a set S is a family of disjoint nonempty subsets of S, called *blocks*, whose union is S. Partitions are ordered by refinement: $\pi \leqq \sigma$ if every block of π is contained in one block of σ. The letter O denotes the partition where each block has one element. For the statement of the Möbius Inversion Formula we refer to the author's paper; little beyond the statement is needed. Bracketed statements and formulas denote partial results. Some details are omitted, but enough are given so that the full proof can be reconstructed.

3. **Generating functions.** A *function* is a function from a finite set of integers to the positive integers N. We associate to every function $f: S \rightarrow N$ a formal power series in the variables x_j^i, $i \in S$, $1 \leqq j < \infty$, as follows.

Let

$$M(f) = \prod_{i \in S} x_{f(i)}^{i};$$

if A is a set of functions, then let

$$M(A) = \sum_{f \in A} M(f).$$

Call $M(A)$ the *generating function* of the set A.

(1) Let C and B be the sets of functions from S and T to N, respectively, where $S\cap T=\varnothing$, and let A be their product; then $M(A)=M(B)M(C)$.

If A and B are any sets of functions, then $M(A\cup B)=M(A)+M(B)$.

(2) Let $U(S)$ be the set of all functions on S each of which takes only one value, then

$$M(U(S)) = \sum_{j\in N}\left(\prod_{i\in S} x_j^i\right).$$

(3) Let π be a partition of S, and let $B(\pi)$ be the generating function of the set of functions $f\colon S\to N$ such that every restriction $f|B$ takes only one value, for every block B of π. From (1) and (2) we infer

$$B(\pi) = \prod_{B\in\pi} M(U(B)).$$

For example, if $\pi=0$, then $B(O)$ is the generating function of the set of all functions on S, and $B(O)=\prod_{i\in S}(x_1^i+x_2^i+x_3^i+\cdots)$.

(4) Let A be the set of one-to-one functions; then

$$M(A) = \sum\prod_{i\in S} x_{k_i}^i,$$

where the sum ranges over all products, each product appearing only once, and the factors $x_{k_i}^i$, $i\in S$, in each product being distinct.

For example, if $S=\{i,\ 1\leqq i\leqq n\}$, then $M(A)=\sum x_{k_1}^1x_{k_2}^2\cdots x_{k_n}^n$, where the sum ranges over all distinct sequences $(k_1, k_2, \cdots, k_n)$ of distinct integers.

The formal power series $M(A)$ are "polarized forms" of the elementary symmetric functions.

4. Möbius inversion. Let G be a group of permutations of the set S, and H be a subgroup of G. Associate to H a partition π of S as follows: two elements $a, b\in S$ belong to the same block of π, whenever there is a permutation $p\in H$ such that $p(a)=b$. The partition π is called the *period of* the subgroup H. A partition π which is the period of some subgroup H of G is called a *period* of the group action (G, S). The set of all periods is a lattice $P(G, S)$. Given a function $f\colon S\to N$, the period of the group of all $p\in G$ s.t. $f(p(s))=f(s)$ for all $s\in S$ is called the *G-period of f*. For example, if G is the group of all permutations of S, then the G-period of f is called the *co-image* of f (cf. Mitchell, also called the *kernel*).

(1) Let $\pi\in P(G, S)$, let $B(\pi)$ be the generating function of the set of functions $f\colon S\to N$ whose co-image is some partition σ such that

$\sigma \geqq \pi$, and let $B'(\pi)$ be that of the set of functions whose G-period is some partition τ such that $\tau \geqq \pi$. Then $B'(\pi) = B(\pi)$. For, if τ and σ are respectively the G-period and the co-image of f, then $\tau \leqq \sigma$.

For $\pi \in P(G, S)$, let $A(G, \pi)$ be the generating function of the set of functions whose G-period is π. Clearly

$$B(\tau) = B'(\tau) = \sum_{\pi \geqq \tau} A(G, \pi), \qquad \tau, \pi \in P(G, S). \tag{2}$$

By Möbius Inversion in the lattice $P(G, S)$ we infer $A(G, \tau) = \sum_{\pi \geqq \tau} \mu(\tau, \pi) B(\pi)$, where μ is the Möbius function of the lattice $P(G, S)$. Setting $\tau = 0$, we obtain

$$A(G, 0) = \sum_{\pi \in P(G,S)} \mu(0, \pi) B(\pi). \tag{3}$$

5. **Main result.** Let $A(\pi)$ be the generating function of the set of functions whose co-image is π. If F is a formal power series in the variables x_j^i, $1 \leqq i \leqq n$, $1 \leqq j < \infty$, let $T_k(F)$ be the polynomial obtained by setting to zero all variables x_j^i for $j > k$. Let $S = \{I: 1 \leqq i \leqq n\}$ and let $A_k(G, \pi) = T_k(A(G, \pi))$. Let

$$\boldsymbol{A}(G, \pi) = (A_1(G, \pi), A_2(G, \pi), A_3(G, \pi), \cdots). \tag{1}$$

The sequence $\boldsymbol{A}(\pi)$ is similarly defined, and belongs to the Standard Baxter Algebra generated by the free sequences $x^1 = (x_1^1, x_2^1, x_3^1, \cdots)$, $x^2, \cdots, x^n$. In fact, an explicit expression can be given for $\boldsymbol{A}(\pi)$: Let $B_1, B_2, \cdots, B_k$ be the blocks of π, and set $x(B) = \prod_{i \in B} x^i$. Then,

$$\boldsymbol{A}(\pi) = \sum_p P(x(B_{p1})P(x(B_{p2}) \cdots P(x(B_{pk})) \cdots)), \tag{2}$$

the sum ranging over all permutations p of $\{1, 2, \cdots, k\}$. (The verification is easy.) For the special case $\pi = 0$ we obtain

$$\boldsymbol{A}(0) = \sum_p P(x^{p1}P(x^{p2}P(\cdots Px^{pn}) \cdots)), \tag{3}$$

where $A(0)$ is the generating function of the set of all one-to-one functions (cf. §3, (4)). Similarly, set $\boldsymbol{B}(\pi) = (B_1(\pi), B_2(\pi), B_3(\pi), \cdots)$, where $B_k(\pi) = T_k B(\pi)$. Then $\boldsymbol{B}(\pi)$ belongs to the Standard Baxter Algebra (cf. §3, (3)):

$$\boldsymbol{B}(\pi) = Px(B_1)Px(B_2) \cdots Px(B_k). \tag{4}$$

Let σ be a partition of S, and let $\bar{\sigma} = \sup\{\pi: \pi \leqq \sigma, \pi \in P(G, S)\}$. Then $\sigma \rightarrow \bar{\sigma}$ is a closure relation on the dual lattice of $P(G, S)$. If the co-image of f is σ, the G-period of f is $\bar{\sigma}$. Hence, $A(G, \pi) = \bigcup_{\{\sigma: \bar{\sigma} = \pi\}} A(\sigma)$, $\pi \in P(G, S)$. Applying the operator T_k to both sides,

$$A_k(G, \pi) = \sum_{\{\sigma:\, \bar{\sigma}=\pi\}} A_k(\sigma) \tag{5}$$

whence we conclude that $A(G, \pi) = \sum_{\{\sigma:\, \bar{\sigma}=\pi\}} A(\sigma)$ is an element of the Standard Baxter Algebra.

To §4, (3), apply T_k, obtaining

$$A_k(G, 0) = \sum_{\pi \in P(G,\pi)} \mu(0, \pi) B_k(\pi); \tag{6}$$

this in turn gives the following identity, in the Standard Baxter algebra:

$$A(G, 0) = \sum_{\pi \in P(G,S)} \mu(0, \pi) B(\pi) = \sum_{\{\sigma:\, \bar{\sigma}=0\}} A(\sigma). \tag{7}$$

In virtue of the Theorem proved in I, identity (7) expresses the equality of two expressions involving only a Baxter operator P and generic variables $x^1, x^2, \cdots, x^n$, and is therefore valid in any Baxter algebra.

Example 1. Let G be the symmetric group. Then $\boldsymbol{A}$ and $\boldsymbol{B}$ in (7) are given by (3) and (4), and we obtain the classical Bohnenblust-Baxter-Spitzer formula

$$\begin{aligned} &\sum_p P(x^{p_1} P(x^{p_2} \cdots (P x^{p_n}) \cdots)) \\ &\qquad = \sum_\pi \mu\,(0, \pi) Px(B_1) Px(B_2) \cdots Px(B_k), \end{aligned} \tag{8}$$

where the B_i are the blocks of π, and the sum ranges over all partitions of $\{1, 2, \cdots, n\}$. The Möbius function of the lattice of partitions is known (cf. the author's paper), it is

$$\mu(0, \pi) = \prod_{B \in \pi} (-1)^{\nu(B)-1} (\nu(B) - 1)!,$$

where $\nu(B)$ denotes the size of the set B. Replacing in (8), the classical version is obtained.

Example 2. Let G be a cyclic group of order n acting transitively on $S = \{1, 2, \cdots, n\}$. Then $P(G, S)$ is isomorphic to the lattice of divisors of the integer n, and $\mu(0, \pi) = \bar{\mu}(n/k)$ if π is a period with k blocks, where $\bar{\mu}$ is the classical Möbius function. Formula (7) is trivial if n is a prime, and is significant when n has a large number of divisors.

Other interesting possibilities arise when G is a product of symmetric groups, and when G is the alternating group. The reader may enjoy working them out by himself.

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