STURM COMPARISON THEOREMS FOR ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS

BY J. B. DIAZ AND JOYCE R. McLAUGHLIN

Communicated September 30, 1968

1. **Introduction.** Comparison theorems for differential equations have attracted the attention of mathematicians for a long time. The first results of this nature, concerning linear, ordinary, second order, selfadjoint, differential equations $(ru')'+pu=0$, go back to Sturm [1]. Further developments along this line are due to M. Picone [2], and M. Bôcher [3], [4]; an account of this theory is to be found in the book of E. L. Ince [5]. The adjective "Sturm," in the title of this research announcement, is meant to refer, in a general way, to results similar to Sturm's, not only for single ordinary differential equations, but also for single partial differential equations and for systems of ordinary or partial differential equations.

Most of the "Sturm comparison theorems" announced here can be described roughly as follows: the hypothesis is that a certain nontrivial function is a solution of a given equation, and is zero on the boundary of a bounded, open, connected set, while the conclusion asserts that a solution of a related equation has a zero in the open set under consideration. In other Sturm comparison theorems announced here, the conclusion remains the same, while the hypothesis is now that a certain nontrivial function satisfies an integral inequality, and is zero on the boundary of a bounded, open, connected set. This integral inequality hypothesis is suggested by the work of W. Leighton [6]. The proof of both kinds of Sturm's theorems depends, in an essential manner, on identities similar to that of the Picone identity in the classical Sturm theory for ordinary (M. Picone [2]) and partial differential equations (M. Picone [7]).

2. **Ordinary differential equations.** In seeking to prove a "Sturm comparison theorem, for a single equation or a system of equations, under the weakest possible hypotheses," the following theorem was obtained.

THEOREM 1. *Suppose*

(1) *z and u^q, for $q=1, \cdots, m$, are differentiable vector valued functions with real components on $-\infty<a\leqq x\leqq b<+\infty$, $a\neq b$;*

(2) *r, $\bar{r}$, p, $\bar{p}$ are symmetric, m by m matrices with real valued components defined on $a\leqq x\leqq b$; $\bar{r}$ is positive definite for $a\leqq x\leqq b$; z and u^q*

are solutions of the systems of differential equations $(rz')'+pz=0$, $[\bar{r}(u^q)']'+\bar{p}u^q=0$, *respectively, for* $a\leqq x\leqq b$, $q=1,\cdots,m$;

(3) *the derivatives* $(rz')'$ *and* $[\bar{r}(u^q)']'$ *are assumed to exist for* $a\leqq x\leqq b$; *there exist positive constants* R *and* P, *with* $\|\bar{r}^{-1}\|\leqq R$ *and* $\|\bar{p}\|\leqq P$ *for* $a<x<b$;

(4) $\{u^q\}_{q=1}^m$ *is a basis of a conjugate family*;

(5) $\bar{p}-p$ *and* $r-\bar{r}$ *are nonnegative definite throughout* $a<x<b$;

(6) *either*

(6_1) $\bar{p}-p$ *is positive definite for some* x *in* $a<x<b$, *or*

(6_2) $z(r-\bar{r})z>0$ *for some* x *in* $a<x<b$, *or*

(6_3) z *is not a linear combination of the* $\{u^q\}_{q=1}^m$ *throughout* $a<x<b$;

(7) $z(a)=z(b)=0$, $z\neq 0$ *throughout* $a<x<b$.

Then at least one nontrivial linear combination of $\{u^q\}_{q=1}^m$ *has a zero in the open interval* $a<x<b$.

(*Explanation.* In the case of a single equation, $m=1$, every nontrivial solution is "a basis of a conjugate family".) The proof of this theorem is based upon a uniqueness theorem for the system of equations, using *only* hypotheses (1), (2), (3), and also upon the following elementary

LEMMA. *Suppose*

(1) $w(x)$ *is a real valued continuous function, defined on the closed interval* $a\leqq x\leqq b$;

(2) *the derivative* $w'(x)$ *exists in the open interval* $a<x<b$;

(3) $w'(x)\geqq 0$ *in the open interval* $a<x<b$, *and* $w'(x)>0$ *for at least one* x *with* $a<x<b$.

Then $w(b)>w(a)$.

This lemma, whose proof uses only the elementary mean value theorem of the differential calculus, makes it possible (together with the uniqueness theorem mentioned) to avoid the usual continuity hypotheses (on both solutions and coefficients) involved in employing Riemann integration (see, e.g., Ince [5]) in the proof.

3. **Partial differential equations.** In seeking to obtain corresponding "Sturm comparison theorems, for partial differential equations, with the weakest possible hypotheses," one is led to consider the validity of the following

PSEUDO-LEMMA. *Suppose*

(1) D *is a nonempty, open, connected, bounded set in* E^n, n *dimensional real Euclidean space, with boundary* B; *an element* x *in* E^n *will be denoted by* $x=(x_1,\cdots,x_n)$;

(2) $f=(f_1, \cdots, f_n)$ *is a vector valued function, with real components, defined and continuous on* $D+B$;

(3) *the partial derivative* $\partial f_i/\partial x_i$ *exists in* D, *for* $i=1, \cdots, n$; $\operatorname{div} f=\sum_{i=1}^n \partial f_i/\partial x_i \geqq 0$ *in* D, *while* $\operatorname{div} f \not\equiv 0$ *in* D.

Then $f \not\equiv 0$ *on* B.

However, this pseudo-lemma is not true, as the following example (which may be deduced fiom a well-known example due to Peano [8]) shows: Let the function g be defined for all x_1, x_2 by $g(x_1, x_2)=x_1x_2(x_1^2-x_2^2)(-x_1^2-x_2^2+2)$; while $h(x_1, x_2)=x_1x_2(x_1^2-x_2^2)\cdot(x_1^2+x_2^2)^{-1}$ for $(x_1, x_2)\neq(0, 0)$, and $h(0,0)=0$. Next, define the vector $f=(f_1, \cdots, f_n)$ to be

$$f_1=\left[-\frac{\partial g}{\partial x_2}+\frac{\partial h}{\partial x_2}\right]\prod_{i=3}^n x_i(1-x_i),$$

$$f_2=\left[\frac{\partial g}{\partial x_1}-\frac{\partial h}{\partial x_1}\right]\prod_{i=3}^n x_i(1-x_i);$$

$f_j\equiv 0$ for $j=3, \cdots, m$. One has $\operatorname{div} f=0$ for $(x_1, \cdots, x_n)\neq(0, 0, x_3, \cdots, x_n)$, while $\operatorname{div} f=2\cdot\prod_{i=3}^n x_i(1-x_i)$ for $(x_1, \cdots, x_n)=(0, 0, x_3, \cdots, x_n)$. If

$$D=\left\{(x_1, \cdots, x_n)\,\middle|\,x_1^2+x_2^2<1; 0<x_i<1, i=3, \cdots, n\right\},$$

then $f_k=0$ on B for $k=1, \cdots, n$, so that f satisfies all the hypotheses, but not the conclusion, of the pseudo-lemma.

Consequently, a lemma with stronger hypotheses is essential, in the case of partial differential equations, to obtain a Sturm comparison theorem. One such lemma may be obtained by applying Green's theorem to a sequence of "regular" domains, which are contained, together with their boundaries, in D. Using this valid lemma, together with a result of E. Hopf [9], the following was obtained:

THEOREM 2. *Suppose*

(1) D *is a nonempty open, bounded, "regular," connected set in* E^n, *with boundary* B;

(2) *there exists a sequence of open, "regular" sets* D_j, *with boundary* B_j, *such that* $D_j+B_j\subset D$, *and, for each* $x\in D-D_j$, *the distance* $d(x, B)<1/j$, *while there is a positive number* S *such that the area* $S_j=\int_{B_j}ds<S$ *for all* $j=1, 2, \cdots$;

(3) *for every* $\bar{x}\in B$ *there exists* $\bar{\bar{x}}\in D$ *and* $\delta>0$ *such that ("sphere property") the sphere* $S(\bar{\bar{x}}, \delta)=\{x\,|\,d(x, \bar{\bar{x}})<\delta\}\subset D$ *and* $d(\bar{x}, \bar{\bar{x}})=\delta$;

(4) D_1 *is an open set in* E^n, *with* $D+B\subset D_1$;

(5) u, v *are real valued functions in* $C^2(D_1)$;

(6) (a_{ij}) and (a_{ij}^*), for $i, j=1, \cdots, n$, are symmetric, positive definite matrices, whose components are real valued and continuous, together with their first partial derivatives $\partial/\partial x_i(a_{ij})$, $\partial/\partial x_i(a_{ij}^*)$, for $i, j=1, \cdots, n$, in D_1; b and b^* are real valued continuous functions in D_1, with $b^* \geqq 0$ on $D+B$;

(7) u and v satisfy, in D_1, the elliptic selfadjoint partial differential equation

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + bu = 0$$

and

$$L^*v = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}^*\frac{\partial v}{\partial x_j}\right) + b^*v = 0;$$

(8) $b^*-b \geqq 0$ in D, and $(a_{ij}-a_{ij}^*)$ is nonnegative definite in D;

(9) either

(9_1) $b^*-b>0$ for some $x \in D$, or

(9_2)

$$\sum_{i,j=1}^{n} (a_{ij} - a_{ij}^*)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} > 0$$

for some $x \in D$, or

(9_3) u is not a constant multiple of v throughout D;

(10) $u=0$ in B, but $u>0$ in D.

Then v must have a zero in D.

There is also a theorem analogous to Theorem 2, with the same conclusion, and the same hypotheses (1)–(5) and (10), while hypotheses (6)–(9) are replaced by

(6′) (a_{ij}^*), for $i, j=1, \cdots, n$, and b^* satisfy the conditions of hypothesis (6) of Theorem 2;

(7′) $L^*v=0$ for $x \in D_1$;

$$(8') \qquad \int_D \left[-\sum_{i,j=1}^{n} a_{ij}^*\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} + b^*u^2\right] dx_1, \cdots, dx_n \geqq 0;$$

(9′) either

$(9_1')$ the strict inequality sign holds in (8′), or

$(9_2')$ u is not a constant multiple of v throughout D.

It is expected that complete proofs of the announced and related results will appear elsewhere.

References

1. J. C. F. Sturm, J. de Math. **1** (1836), 106.
2. Mauro Picone, Ann. Scuola Norm. Sup. Pisa **11** (1909), 1.
3. Maxime Bôcher, *Boundary problems in one dimension*, Proc. 5th Int. Congress Math., I, Cambridge, 1912, pp. 163–195.
4. ———, *Leçons sur les methodes de Strum*, Gauthier-Villars, Paris, 1917.
5. E. L. Ince, *Ordinary differential equations*, Longmans, London, 1927; Dover, N.Y., 1956.
6. Walter Leighton, *A substitute for the Picone formula*, Bull. Amer. Math. Soc. **55** (1949), 325–328.
7. Mauro Picone, Rend. Acc. Lincei **20** (1) (1911), 213–219.
8. G. Peano (see E. W. Hobson, *The theory of functions of a real variable*. I, reprint, Dover, N.Y., 1957, p. 429).
9. E. Hopf, *A remark on linear elliptic differential equations of second order*, Proc. Amer. Math. Soc. **3** (1952), 791–793.

Rensselaer Polytechnic Institute, Troy, New York 12181