ON THE STABLE DIFFEOMORPHISM OF HOMOTOPY SPHERES IN THE STABLE RANGE, $n<2p$

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Communicated by William Browder, September, 20, 1968

1. Introduction and statement of results. Let Θ_n^{p+1} denote the subgroup of the Kervaire-Milnor group θ_n of those homotopy n-spheres imbedding with trivial normal bundle in euclidean $(n+p+1)$-space $(n<2p)$. It is known that every homotopy n-sphere Σ^n imbeds in $(n+p+1)$-space with normal bundle independent of the imbedding provided, $n<2p$, [8]. Let $\Omega_{n,p}$ denote the quotient group θ_n/Θ_n^{p+1}.

It has been proved both by the author and R. DeSapio [3] that the order of $\Omega_{n,p}$, after identifying elements with their inverses, is just the number of diffeomorphically distinct products $\Sigma^n\times S^p$. It is shown in [3] that the *stable range* $n<2p$ is not necessary for the theorem. However, *it is crucial for all our own work on* $\Omega_{n,p}$. Indeed, it is in the stable range that the calculation of $\Omega_{n,p}$ is reducible to an effectively computable homotopy question. Further results on properties of $\Omega_{n,p}$, and in particular its relation to the determination of the number of smooth structures on $S^n\times S^p$, can be found in the very interesting work of DeSapio [3], [4] and [5].

From results of [8] it is immediate that $\Omega_{n,p}=0$ for $p\geqq n-3$ or $n\leqq 15$, $n<2p$ and $\Omega_{16,12}=Z_2$; the following theorems are extensions of these results for the stable range $n<2p$.

THEOREM 1.1. (i) $\Omega_{n,p}=0$ *if* $p\geqq n-7$ *and* $n\not\equiv 0, 1 \pmod 8$.
(ii) $\Omega_{n,n-4}=Z_2$ *for* $n=16, 32$.
(iii) $\Omega_{17,10}=Z_2$; $\Omega_{n,p}=0$ *if* $p\geqq n-6$ *and* $n\equiv 1 \pmod 8$.

Parts (ii) and (iii) show that (i) is best possible. However, we can also show

THEOREM 1.2. $\Omega_{n,n-13}=0$ *if* $n\equiv 4, 5 \pmod 8$.

Therefore, (i) of Theorem 1.1 is by no means the final answer. The table below gives our results for $n\leqq 20$.

Letting $\phi_n^{p+1}\colon \theta_n\to\pi_{n-1}(\mathrm{SO}(p+1))$ be the characteristic homomorphism of [8] we have

THEOREM 1.3. $\Omega_{n,p}=\operatorname{im}\phi_n^{p+1}$.

[1] This research was partially supported by NSF grants GP-8888 and GP-7952X.

This result is easily proved and is the basis for reduction of the stable diffeomorphism question in the stable range to a homotopy problem.

TABLE I. THE GROUP $\Omega_{n,p}$, $n \leqq 20$

$n \backslash p$	9	10	11	12	$p \geqq 13$
16	Z_2	Z_2	Z_2	Z_2	0
17	Z_2	Z_2	0	0	0
18	/	Z_{2d}	0	0	0
19	/	Z_{2d}	Z_{2d}	0	0
20	/	/	$Z_{2d}^{(3)}$	0	0

Z_{2d} denotes Z_2 (if $d=1$) or the zero group (if $d=0$) and $Z_{2d}^{(3)}$ denotes the direct sum of 3 copies of Z_{2d}. A square with a slash through it is out of the stable range.

2. **Indication of proofs.** It is easy to show from Theorem 1.3 that

$$\Omega_{n,p} \subseteq \ker i_{n-1}^{p+1} \cap \ker J_{n-1}^{p+1} \tag{2.1}$$

for $n<2p$, where $i_{n-1}^{p+1}\colon \pi_{n-1}(\mathrm{SO}(p+1)) \to \pi_{n-1}(\mathrm{SO})$ is induced from the inclusion $\mathrm{SO}(p+1) \subseteq \mathrm{SO}$ and J_{n-1}^{p+1} is the J-homomorphism in the PSH diagram below.

$$\begin{tikzcd}
 & \pi_{n-1}(\mathrm{SO}(p+1)) \arrow[r, "i_{n-1}^{p+1}"] \arrow[dd, "J_{n-1}^{p+1}"] & \pi_{n-1}(\mathrm{SO}(p+2)) \arrow[dd, "J_{n-1}^{p+2}"] \arrow[dr, "p_*"] & \\
\pi_n(S^{p+1}) \arrow[ur, "\partial"] \arrow[dr, "P"'] & & & \pi_{n-1}(S^{p+1}) \\
 & \pi_{n+p}(S^{p+1}) \arrow[r, "S"'] & \pi_{n+p+1}(S^{p+2}) \arrow[ur, "H"'] &
\end{tikzcd} \tag{2.2}$$

See [10] for definitions of J and H; S is just suspension; the top sequence is part of the fibre-homotopy sequence for the fibering $\mathrm{SO}(p+2) \xrightarrow{p} S^{p+1}$ while the lower sequence is due to G. Whitehead and is exact for $n<2p$; the Diagram 2.2 commutes up to sign.

From the metastable splitting of $\pi_i(\mathrm{SO}(n))$ due to Barratt and Mahowald [2] it follows that

$$\ker i_{n-1}^{p+1} = \pi_n(V_{2(p+1),p+1}) \tag{2.3}$$

for $n<2p$ and $p\geqq 12$. Theorem 2 follows directly from 2.1, 2.3 and results of [7].

In [8] it is proved that the monomorphism of 2.1 is epi if $n\not\equiv 2$ (mod 4). This fact and 2.2 form the basis for proving (ii) of Theorem 1.1. It is clear from the PSH diagram and tables of Kervaire [9] that $\Omega_{n,n-4}=Z_2$ for $n\equiv 0$ (mod 8) iff $P(\alpha_n)=0$, where $P(\alpha_n)$ is the Whitehead product of the generator α_n of $\pi_n(S^{n-3})=Z_{24}$ with that of $\pi_{n-3}(S^{n-3})$. Since it is known that $P(\alpha_n)=0$ for $n=16$, 32, (ii) of Theorem 1.1 is proved. However, $P(\alpha_{24})\neq 0$.[2]

It is known [9, p. 168, II.10] that the sequence

$$(2.4)\qquad 0\rightarrow \pi_{8s+1}(V_{m,m-8s+i})\rightarrow \pi_{8s}(\mathrm{SO}(8s-i))\rightarrow \pi_{8s}(\mathrm{SO})\rightarrow 0$$

is exact if $i\leqq 6$, $s\geqq 2$ and m is large enough. Here $V_{n,r}$ denotes the real Stiefel manifold of r-frames in n-space. In [8] it is proved that the sequence

$$(2.5)\qquad 0\rightarrow bP_{n+1}\rightarrow \Theta_n^{p+1}\rightarrow \operatorname{cok} J_n^{p+1}\rightarrow 0 \qquad n\not\equiv 2\ (\mathrm{mod}\ 4)$$

is exact in the stable range if $n>4$ and $p\geqq 2$; bP_{n+1} denotes the group of homotopy n-spheres which bound π-manifolds. Using tables in [6] and [9], (iii) of Theorem 1.1 is established via 2.2, 2.4 and 2.5.

Part (i) of Theorem 1.1 is proved by "pushing back the J-homomorphism through successive stages of consecutive PSH diagrams" establishing monomorphisms for appropriate pieces of the J-homomorphism at each stage (there are sometimes obstructions to the entire J_{n-1}^{p+1} being a monomorphism). Extensive use is made of calculations of [6], [7], [9], [12] and [13]. The sequence 2.4 and others like it, plus 2.5, are used throughout.

The results in Table I are proved using the above-mentioned techniques together with results on the order of θ_n ($n<20$) in [11]. The simple fact that $\Theta_n^{p+1}\subseteq\Theta_n^{p+2}$ in the stable range is also inportant.

In conclusion, we should perhaps mention that our original approach to the solution of the stable diffeomorphism question in the range $n<2p$ made use of the notion of h-enclosability [1]. However, the present approach has since been seen to be simpler.

Added in proof. The author has completed calculations for $n\leq 28$, $n<2p$. The only additional nonzero groups (except possibly $\Omega 24$, 13) are $\Omega 18$, 10 and $\Omega 19$, 10, each of which is cyclic of order two.

[2] The author is grateful to M. Mahowald for the cases $n=24$, 32.

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