ON THE STABLE DIFFEOMORPHISM OF HOMOTOPY SPHERES IN THE STABLE RANGE, \( n < 2p \)

BY P. L. ANTONELLI

Communicated by William Browder, September 20, 1968

1. Introduction and statement of results. Let \( \Theta_{n+1} \) denote the subgroup of the Kervaire-Milnor group \( \theta_n \) of those homotopy \( n \)-spheres imbedding with trivial normal bundle in euclidean \( (n+p+1) \)-space \( (n < 2p) \). It is known that every homotopy \( n \)-sphere \( \Sigma^n \) imbeds in \( (n+p+1) \)-space with normal bundle independent of the imbedding provided, \( n < 2p \), [8]. Let \( \Omega_{n,p} \) denote the quotient group \( \theta_n/\Theta_{n+1} \).

It has been proved both by the author and R. DeSapio [3] that the order of \( \Omega_{n,p} \), after identifying elements with their inverses, is just the number of diffeomorphically distinct products \( S^n \times S^p \). It is shown in [3] that the stable range \( n < 2p \) is not necessary for the theorem. However, it is crucial for all our own work on \( \Omega_{n,p} \). Indeed, it is in the stable range that the calculation of \( \Omega_{n,p} \) is reducible to an effectively computable homotopy question. Further results on properties of \( \Omega_{n,p} \), and in particular its relation to the determination of the number of smooth structures on \( S^n \times S^p \), can be found in the very interesting work of DeSapio [3], [4] and [5].

From results of [8] it is immediate that \( \Omega_{n,p} = 0 \) for \( p \geq n-3 \) or \( n \leq 15, n < 2p \) and \( \Omega_{16,12} = Z_2 \); the following theorems are extensions of these results for the stable range \( n < 2p \).

**Theorem 1.1.**

(i) \( \Omega_{n,p} = 0 \) if \( p \geq n-7 \) and \( n \equiv 0, 1 \) \( (\text{mod } 8) \).

(ii) \( \Omega_{n,n-4} = Z_2 \) for \( n = 16, 32 \).

(iii) \( \Omega_{17,10} = Z_2; \Omega_{n,p} = 0 \) if \( p \geq n-6 \) and \( n \equiv 1 \) \( (\text{mod } 8) \).

Parts (ii) and (iii) show that (i) is best possible. However, we can also show

**Theorem 1.2.** \( \Omega_{n,n-12} = 0 \) if \( n = 4, 5 \) \( (\text{mod } 8) \).

Therefore, (i) of Theorem 1.1 is by no means the final answer. The table below gives our results for \( n \leq 20 \).

Letting \( \phi_{n+1}: \theta_n \rightarrow \pi_{n-1}(SO(p+1)) \) be the characteristic homomorphism of [8] we have

**Theorem 1.3.** \( \Omega_{n,p} = \text{im } \phi_{n+1} \).

---

1 This research was partially supported by NSF grants GP-8888 and GP-7952X.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
This result is easily proved and is the basis for reduction of the stable diffeomorphism question in the stable range to a homotopy problem.

<table>
<thead>
<tr>
<th>Table I. The Group $\Omega_{n,p}$, $n \leq 20$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n \backslash p$</td>
</tr>
<tr>
<td>------------------</td>
</tr>
<tr>
<td>16</td>
</tr>
<tr>
<td>17</td>
</tr>
<tr>
<td>18</td>
</tr>
<tr>
<td>19</td>
</tr>
<tr>
<td>20</td>
</tr>
</tbody>
</table>

$\mathbb{Z}_d$ denotes $\mathbb{Z}_1$ (if $d = 1$) or the zero group (if $d = 0$) and $\mathbb{Z}_d^{(3)}$ denotes the direct sum of 3 copies of $\mathbb{Z}_d$. A square with a slash through it is out of the stable range.

2. Indication of proofs. It is easy to show from Theorem 1.3 that

$$\Omega_{n,p} \subseteq \ker \pi_{n-1}^{p+1} \cap \ker J_{n-1}^{p+1}$$

for $n < 2p$, where $\pi_{n-1}^{p+1}: \pi_{n-1}(\text{SO}(p+1)) \to \pi_{n-1}(\text{SO})$ is induced from the inclusion $\text{SO}(p+1) \subseteq \text{SO}$ and $J_{n-1}^{p+1}$ is the $J$-homomorphism in the PSH diagram below.

![Diagram 2.2](image)

See [10] for definitions of $J$ and $H$; $S$ is just suspension; the top sequence is part of the fibre-homotopy sequence for the fibering $\text{SO}(p+2) \to \text{S}^{p+1}$ while the lower sequence is due to G. Whitehead and is exact for $n < 2p$; the Diagram 2.2 commutes up to sign.

From the metastable splitting of $\pi_{n}(\text{SO}(n))$ due to Barratt and Mahowald [2] it follows that

$$\ker \pi_{n-1}^{p+1} = \pi_{n}(\text{V}_{n}(p+1)).$$
for \( n < 2p \) and \( p \geq 12 \). Theorem 2 follows directly from 2.1, 2.3 and results of [7].

In [8] it is proved that the monomorphism of 2.1 is epi if \( n \not\equiv 2 \) (mod 4). This fact and 2.2 form the basis for proving (ii) of Theorem 1.1. It is clear from the PSH diagram and tables of Kervaire [9] that \( \Omega_n, n-4 = Z_2 \) for \( n \equiv 0 \) (mod 8) iff \( P(\alpha_n) = 0 \), where \( P(\alpha_n) \) is the Whitehead product of the generator \( \alpha_n \) of \( \pi_n(S^{n-3}) = Z_2 \) with that of \( \pi_{n-3}(S^{n-3}) \). Since it is known that \( P(\alpha_n) = 0 \) for \( n = 16, 32 \), (ii) of Theorem 1.1 is proved. However, \( P(\alpha_{24}) \neq 0 \).

It is known [9, p. 168, II.10] that the sequence

\[
(2.4) \quad 0 \to \pi_{8s+1}(V_{n,m-s+1}) \to \pi_{8s}(SO(8s-i)) \to \pi_{8s}(SO) \to 0
\]

is exact if \( i \leq 6, s \geq 2 \) and \( m \) is large enough. Here \( V_{n,r} \) denotes the real Stiefel manifold of \( r \)-frames in \( n \)-space. In [8] it is proved that the sequence

\[
(2.5) \quad 0 \to bP_{n+1} \to \Theta_n^{p+1} \to \text{cok} J_n^{p+1} \to 0 \quad n \not\equiv 2 \; (\text{mod} \; 4)
\]

is exact in the stable range if \( n > 4 \) and \( p \geq 2 \); \( bP_{n+1} \) denotes the group of homotopy \( n \)-spheres which bound \( \pi \)-manifolds. Using tables in [6] and [9], (iii) of Theorem 1.1 is established via 2.2, 2.4 and 2.5.

Part (i) of Theorem 1.1 is proved by "pushing back the \( J \)-homomorphism through successive stages of consecutive PSH diagrams" establishing monomorphisms for appropriate pieces of the \( J \)-homomorphism at each stage (there are sometimes obstructions to the entire \( J_n^{p+1} \) being a monomorphism). Extensive use is made of calculations of [6], [7], [9], [12] and [13]. The sequence 2.4 and others like it, plus 2.5, are used throughout.

The results in Table I are proved using the above-mentioned techniques together with results on the order of \( \theta_n \) (\( n < 20 \)) in [11]. The simple fact that \( \Theta_n^{p+1} \subseteq \Theta_n^{p+2} \) in the stable range is also important.

In conclusion, we should perhaps mention that our original approach to the solution of the stable diffeomorphism question in the range \( n < 2p \) made use of the notion of \( h \)-enclosability [1]. However, the present approach has since been seen to be simpler.

**Added in Proof.** The author has completed calculations for \( n \leq 28, n < 2p \). The only additional nonzero groups (except possibly \( \Omega_{24}, 13 \)) are \( \Omega_{18}, 10 \) and \( \Omega_{19}, 10 \), each of which is cyclic of order two.

---

* The author is grateful to M. Mahowald for the cases \( n = 24, 32 \).
REFERENCES