AN APPLICATION OF ERGODIC THEORY TO THE SOLUTION OF LINEAR FUNCTIONAL EQUATIONS IN BANACH SPACES

BY W. G. DOTSON, JR.[1]

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Suppose E is a Banach space, $T: E \rightarrow E$ is a continuous linear operator, and $f \in E$. Browder and Petryshyn [1] have recently considered the solution of the equation $u - Tu = f$ by means of the Picard iteration $x_{n+1} = Tx_n + f$ which generates the sequence of approximations $\{x_n\}$ defined by

$$x_n = T^n x_0 + \left(\sum_{j=1}^{n} T^{j-1} \right) f. \tag{1}$$

The principal result of [1] is that if T is *asymptotically convergent* (i.e., $\{T^n x\}$ converges (strongly) for each $x \in E$), then the following are true: (a) If f is in the range of $I - T$, then for any $x_0 \in E$ the sequence $\{x_n\}$ defined by equation (1) above converges to a solution u of the equation $u - Tu = f$. (b) If, for some $x_0 \in E$, some subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converges to some $y \in E$, then $y - Ty = f$ and $\{x_n\}$ converges to y. (c) If E is reflexive and $\{x_n\}$ is bounded, then $\{x_n\}$ converges to a solution of the equation $u - Tu = f$. We will generalize this result by subsuming it as a special case of a (new) mean ergodic theorem for affine mappings. Our ergodic theorem is an extension of the mean ergodic theorem of Eberlein [3] for semigroups of linear operators. As another special case, we find that an iterative solution of $u - Tu = f$ using Cesàro means of the Picard iterates is possible under conditions on T which are considerably weaker than asymptotic convergence. Statement (b) above will be strengthened by showing that if $\{x_n\}$ has a subsequence $\{x_{n_j}\}$ which converges weakly to some $y \in E$, then $y - Ty = f$ and $\{x_n\}$ converges (strongly) to y. Notice that statement (c) is an immediate corollary of this strengthened version of (b), since closed spheres in a reflexive space are weakly sequentially compact.

Eberlein [3] showed that if G is an ergodic semigroup (of continuous linear operators on E), and if $\{A_\alpha : \alpha \in D\}$ is any system of almost invariant integrals for G, and if $x \in E$ such that the net $\{A_\alpha x : \alpha \in D\}$

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has a weak cluster point $y \in E$, then $y = \lim_\alpha A_\alpha x$ (strongly), and $Ty = y$ for all $T \in G$. The set $\{x: x \in E \text{ and } \lim_\alpha A_\alpha x \text{ exists}\}$ is called the ergodic subspace of E with respect to G (it is independent of which particular system $\{A_\alpha: \alpha \in D\}$ of almost invariant integrals for G is used). We will say that *the ergodic semigroup G is asymptotically convergent* provided the ergodic subspace of E with respect to G is E (i.e. for any system $\{A_\alpha: \alpha \in D\}$ of almost invariant integrals for G, $\lim_\alpha A_\alpha x$ exists for all $x \in E$). Eberlein [3] proved that if $\{A_\alpha: \alpha \in D\}$ is a system of almost invariant integrals for the asymptotically convergent semigroup G, and if $Q: E \to E$ is defined by $Qx = \lim_\alpha A_\alpha x$ for all $x \in E$, then Q is a continuous linear operator, and $Q^2 = Q = UQ = QU$ for any $U \in G$ or $U =$ any A_α.

Suppose $\{A_\alpha: \alpha \in D\}$ is a system of almost invariant integrals for the asymptotically convergent semigroup G, and Q is defined by $Qx = \lim_\alpha A_\alpha x$ for all $x \in E$. A net $\{S_\alpha: \alpha \in D\}$ of continuous linear operators on E will be said to form a *system of companion integrals for* $\{A_\alpha: \alpha \in D\}$ *with respect to* an element $T \in G$ provided: (S1) $(I-T)S_\alpha = S_\alpha(I-T) = I - A_\alpha$ for all $\alpha \in D$, and (S2) $QS_\alpha = \phi(\alpha)Q$ for all $\alpha \in D$, where ϕ is a real-valued function on D such that $\lim_\alpha \phi(\alpha) = +\infty$.

THEOREM 1. *Suppose G is an asymptotically convergent semigroup (of continuous linear operators on E), $T \in G$, and $f \in E$. Suppose $\{A_\alpha: \alpha \in D\}$ is a system of almost invariant integrals for G, and suppose there exists a system $\{S_\alpha: \alpha \in D\}$ of companion integrals for $\{A_\alpha: \alpha \in D\}$ with respect to T. Then, the following are true:* (a) *If f is in the range of $I-T$, then for any $x_0 \in E$ the net $\{x_\alpha: \alpha \in D\}$, defined by $x_\alpha = A_\alpha x_0 + S_\alpha f$, converges to a solution u of the equation $u - Tu = f$.* (b) *If, for some $x_0 \in E$, the net $\{x_\alpha: \alpha \in D\}$ has a (strong) cluster point $y \in E$, then $y - Ty = f$ and $y = \lim_\alpha x_\alpha$ (strongly).* (c) *If, for some $x_0 \in E$, the net $\{x_\alpha: \alpha \in D\}$ is contained in a compact subset of E, then $\{x_\alpha: \alpha \in D\}$ converges to a solution u of the equation $u - Tu = f$.*

We omit the proof of Theorem 1, since it is quite similar to the proof given below for Theorem 2. If one restricts attention to sequences, then the following stronger result can be obtained.

THEOREM 2. *Suppose G is an asymptotically convergent semigroup (of continuous linear operators on E), $T \in G$, and $f \in E$. Suppose the sequence $\{A_n\}$ is a system of almost invariant integrals for G, and suppose there exists a sequence $\{S_n\}$ which forms a system of companion integrals for $\{A_n\}$ with respect to T. Then, the following are true:* (a) *If f is in the range of $I-T$, then for any $x_0 \in E$ the sequence $\{x_n\}$, defined by $x_n = A_n x_0 + S_n f$, converges to a solution u of the equation $u - Tu = f$.* (b) *If, for some $x_0 \in E$, some subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converges* *weakly to some* $y \in E$, *then* $y - Ty = f$ *and* $\{x_n\}$ *converges (strongly) to* y. (c) *If, for some* $x_0 \in E$, *the sequence* $\{x_n\}$ *is contained in a weakly compact subset of* E, *then* $\{x_n\}$ *converges to a solution* u *of the equation* $u - Tu = f$.

PROOF. Strong convergence is denoted by $\to$ and weak convergence by $\rightharpoonup$. (a) There exists $w \in E$ such that $w - Tw = f$; whence $S_n f = S_n(I-T)w = w - A_n w$, so that $S_n f \to w - Qw$. Hence for any $x_0 \in E$ we have $x_n = A_n x_0 + S_n f \to Qx_0 + w - Qw = w + Q(x_0 - w)$ which (since $TQ = Q$) is easily seen to be a solution of $u - Tu = f$. (b) There is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} = A_{n_j} x_0 + S_{n_j} f \rightharpoonup y$. Since $A_n x_0 \to Qx_0$ we have $A_{n_j} x_0 \to Qx_0$. Hence,

$$S_{n_j} f = x_{n_j} - A_{n_j} x_0 \rightharpoonup y - Qx_0.$$

Also we have $Qx_{n_j} = QA_{n_j} x_0 + QS_{n_j} f$, and, using (S2) and $QA_{n_j} = Q$, this becomes

$$Qx_{n_j} = Qx_0 + \phi(n_j) Qf.$$

Q is weakly continuous (since linear and strongly continuous), and so $x_{n_j} \rightharpoonup y$ implies $Qx_{n_j} \rightharpoonup Qy$. Hence we get

$$\phi(n_j) Qf \rightharpoonup Qy - Qx_0.$$

It follows that $\{\phi(n_j) Qf\}$ is bounded, and so, since $\lim_j \phi(n_j) = +\infty$, we must have $Qf = 0$. Using this and (S1) we get

$$(I - T) S_{n_j} f = f - A_{n_j} f \to f - Qf = f.$$

Since $I - T$ is weakly continuous and $S_{n_j} f \rightharpoonup y - Qx_0$, we have

$$(I - T) S_{n_j} f \rightharpoonup (I - T)(y - Qx_0) = y - Ty$$

since $TQ = Q$. Since the weak topology is Hausdorff, it follows that $y - Ty = f$. Part (a) of the present theorem is now applicable, and gives us that $\{x_n\}$ converges to some solution u of $u - Tu = f$. Hence $x_{n_j} \to u$; so that $u = y$ (since $x_{n_j} \rightharpoonup y$) and so $x_n \to y$. Statement (c) is an immediate corollary of statement (b) and the Eberlein-Šmulian theorem that weak compactness in Banach spaces implies weak sequential compactness, cf. [2, p. 430]. Q.E.D.

REMARK. It is clear that u is a solution of the equation $u - Tu = f$ if and only if u is a common fixed point for the semigroup

$$\{(T + f)^n : n = 1, 2, 3, \cdots\}$$

of affine mappings. From this point of view, Theorems 1 and 2 can be considered as extensions of Eberlein's mean ergodic theorem for semigroups of linear operators.

Suppose T is a continuous linear operator on E, and let $G = \{I, T, T^2, \cdots\}$. Suppose $A = [a_{nj}]$ is an infinite real matrix satisfying: (M1) $a_{nj} \geqq 0$ for all n, j, and $a_{nj} = 0$ for $j > n$; (M2) $\sum_{j=1}^{n} a_{nj} = 1$ for each n; (M3) $\lim_n a_{nj} = 0$ for each j. For each positive integer n, define polynomials $a_n(t)$, $s_n(t)$ by

$$a_n(t) = \sum_{j=1}^{n} a_{nj} t^{j-1}, \qquad s_n(t) = (1 - a_n(t))/(1 - t).$$

Condition (M2) insures that $s_n(t)$ is indeed a polynomial. Define $A_n = a_n(T)$, $S_n = s_n(T)$ for each n.

THEOREM 3. *Suppose $G = \{I, T, T^2, \cdots\}$ is an asymptotically convergent semigroup, and $\{A_n\} = \{a_n(T)\}$ is a system of almost invariant integrals for G, and Q is defined by $Qx = \lim_n A_n x$ for all $x \in E$. Then $\{S_n\} = \{s_n(T)\}$ is a system of companion integrals for $\{A_n\}$ with respect to T.*

OUTLINE OF PROOF. $(1-t)s_n(t) = 1 - a_n(t)$ yields (S1) immediately. Dividing out $s_n(t) = (1 - a_n(t))/(1-t)$, one finds that $S_n = s_n(T)$ can be written

$$S_n = \sum_{j=1}^{n-1} \left(1 - \sum_{k=1}^{j} a_{nk}\right) T^{j-1}. \tag{2}$$

Operating on both sides with Q, and using the linearity of Q together with $QT^{j-1} = Q$, one obtains $QS_n = \phi(n)Q$ where $\phi(n) = (\sum_{j=1}^{n} j a_{nj}) - 1$. The proof that $\lim_n \phi(n) = +\infty$ is a fairly straightforward computation using the matrix properties (M1), (M2), (M3). Q.E.D.

In the light of Theorem 3, Theorem 2 now specializes to give immediately the following result.

THEOREM 4. *Suppose $G = \{I, T, T^2, \cdots\}$ is an asymptotically convergent semigroup (T continuous, linear), $\{A_n\} = \{\sum_{j=1}^{n} a_{nj} T^{j-1}\}$ is a system of almost invariant integrals for G, $\{S_n\}$ is defined by equation* (2) *above, and $f \in E$. Then, the following are true*: (a) *If f is in the range of $I - T$, then for any $x_0 \in E$ the sequence $\{x_n\}$, defined by $x_n = A_n x_0 + S_n f$, converges to a solution u of the equation $u - Tu = f$.* (b) *If, for some $x_0 \in E$, $\{x_n\}$ has a subsequence $\{x_{n_j}\}$ which converges weakly to a point $y \in E$, then $y - Ty = f$ and $\{x_n\}$ converges to y.* (c) *If, for some $x_0 \in E$, $\{x_n\}$ is contained in a weakly compact subset of E, then $\{x_n\}$ converges to a solution of the equation $u - Tu = f$.*

The Browder-Petryshyn theorem [1] is obtained as a special case of Theorem 4 by taking A to be the infinite identity matrix, since then we have $a_n(t)=t^{n-1}$, $s_n(t)=(1-t^{n-1})/(1-t)=\sum_{j=1}^{n-1} t^{j-1}$, so that $A_n=T^{n-1}$, $S_n=\sum_{j=1}^{n-1} T^{j-1}$.

DEFINITION. Suppose T is a continuous linear operator on E, $A=[a_{nj}]$ satisfies (M1), (M2), (M3), and $A_n=\sum_{j=1}^{n} a_{nj}T^{j-1}$ for each n. Then: (1) T is said to be *asymptotically A-bounded* provided there exists $M>0$ such that $\|A_n\|\leqq M$ for all n; (2) T is said to be *asymptotically A-regular* provided $\lim_n(TA_nx-A_nx)=0$ for each $x\in E$; (3) T is said to be *asymptotically A-convergent* provided the semigroup $G=\{I, T, T^2, \cdots\}$ is asymptotically convergent with $\{A_n\}$ as a system of almost invariant integrals.

Ordinary asymptotic boundedness, regularity, and convergence for an operator T are obtained by taking A to be the infinite identity matrix in the above definition. The following result is an immediate corollary of Eberlein's mean ergodic theorem [3].

THEOREM 5. *A continuous linear operator T on a Banach space E is asymptotically A-convergent if and only if*: (a) *T is asymptotically A-bounded, and* (b) *T is asymptotically A-regular, and* (c) *$\{A_nx\}$ clusters weakly for each $x\in E$.*

Taking A to be the infinite identity matrix, we get the following immediate corollary.

COROLLARY. *A continuous linear operator T on a Banach space E is asymptotically convergent if and only if*: (a) *T is asymptotically bounded, and* (b) *T is asymptotically regular, and* (c) *$\{T^nx\}$ clusters weakly for each $x\in E$.*

REMARK. In Theorem 5 and its corollary, condition (c) can be omitted if E is reflexive, since (c) then follows from condition (a) and the weak compactness of closed spheres in E. The continuous linear operator $Tx(s)=s\cdot x(s)$, $0\leqq s\leqq 1$, on the nonreflexive Banach space $C[0, 1]$ satisfies (a) and (b) in the above corollary, but is not asymptotically convergent.

Of course, if T is asymptotically A-convergent, then Theorem 4 provides an iterative method for the solution of $u-Tu=f$. Browder and Petryshyn [1] showed that if E is a Hilbert space, and $T: E\rightarrow E$ is a continuous linear selfadjoint operator with $\|T\|\leqq 1$, then T is asymptotically convergent if and only if -1 is not an eigenvalue of T. With $A=C=$ the Cesàro matrix, it is easily seen that the continuous linear operator T on the Hilbert space E is asymptotically C-convergent provided only that $\|T\|\leqq 1$ (or even $\|T^n\|\leqq M$ for all n). We refer to Eberlein [3, pp. 223, 224] for even weaker conditions (in

Banach spaces) under which an operator T is asymptotically C-convergent.

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NORTH CAROLINA STATE UNIVERSITY AT RALEIGH, RALEIGH, NORTH CAROLINA 27607