A PROOF OF JACKSON'S THEOREM

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1. If \( f \) is a continuous function on \([-1,1]\) and \( \omega_f \) its modulus of continuity, then the classical theorem of Jackson [1] states that there exists a sequence of polynomials \( (J_n[f]) \) such that the degree of \( J_n[f] \) is \( \leq n \) and

\[
\max_{|x| \leq 1} |J_n[f](x) - f(x)| \leq C \omega_f \left( \frac{1}{n} \right), \quad n = 1, 2, \ldots
\]

Various direct, but more or less involved proofs of this result are now available in the literature (see [2], [3], [4], [5], [6] and [7]). In [6] it was shown that Legendre polynomials generate approximating polynomials whose deviation from \( f \) on \([-1/4, 1/4]\) is of the order \( \omega_f(1/n) \), as in Jackson's theorem. In [7] this result was extended to a large class of orthogonal polynomials.

The aim of this paper is to give a short and simple direct proof of Jackson's theorem by combining an inequality for positive linear operators which was proved recently by O. Shisha and B. Mond [8], with the ideas developed in [5] and [7].

Let \( T_{2n}(x) = \cos(2n \arccos x) \) be the Chebyshev polynomial of degree \( 2n \), \( \alpha_n = \sin(\pi/4n) \) its smallest positive zero and

\[
R_n(x) = c_n \left( \frac{T_{2n}(x)}{x^2 - \alpha_n^2} \right)^2,
\]

where \( c_n \) is chosen so that \( \int_{-1}^{1} R_n(t) dt = 1 \). Also let

\[
\|g\| = \sup \{ |g(x)| : |x| \leq 1/4 \}.
\]

We shall prove here the following theorem.

**If \( f \) is a continuous function on \([-1/2, 1/2]\), then the polynomial \( K_n[f] \) defined by**

\[
(1) \quad K_n[f](x) = \int_{-1/2}^{1/2} f(t) R_n(t - x) dt
\]

**satisfies the inequality**

\[
(2) \quad \|K_n[f] - f\| \leq 2 \omega_f \left( \frac{1}{n} \right) + 16 \|f\| \frac{1}{n^2}, \quad n = 1, 2, \ldots
\]
In order to obtain from (2) a proof of Jackson's theorem for the interval \([-1/4, 1/4]\), it is sufficient to consider the modified polynomials \(K_n[f]\) defined by \(K_n[f] = f(0) + K_n[f - f(0)]\). Using (2) and elementary properties of the modulus of continuity, we find that for \(n \geq 3\)

\[
\|K_n[f] - f\| \leq 2\omega_f \left(\frac{1}{n}\right) + 16\omega_f \left(\frac{1}{4}\right) \frac{1}{n^2} \leq 4\omega_f \left(\frac{1}{n}\right).
\]

2. In order to simplify the proof of the theorem, we shall first prove the following result.

**Lemma.** For \(n = 1, 2, \ldots\) we have \(\int_{-1}^{1} t^2 R_n(t) dt \leq 1/n^2\).

**Proof of the Lemma.** We have first

\[
\int_{-1}^{1} t^2 R_n(t) dt = \int_{-1}^{1} (1 - t^2)^{-1/2} t^2 R_n(t) dt.
\]

Next, by Gauss quadrature formula based on the zeros of \(T_n\), we have for any polynomial \(P\) of degree \(\leq 4n - 1\)

\[
\int_{-1}^{1} (1 - t^2)^{-1/2} P(t) dt = \frac{\pi}{2n} \sum_{k=1}^{2n} P \left( \cos \frac{2k - 1}{4n} \pi \right)
\]

(see [9, p. 115]). Since \(R_n\) is an even polynomial of degree \(4n - 4\), vanishing at all zeros of \(T_n\) except at \(\alpha_n\) and \(-\alpha_n\), it follows that

\[
\int_{-1}^{1} (1 - t^2)^{-1/2} t^2 R_n(t) dt = \frac{\pi}{n} \alpha_n^2 R_n(\alpha_n)
\]

\[
= \alpha_n^2 \int_{-1}^{1} (1 - t^2)^{-1/2} R_n(t) dt
\]

\[
= \alpha_n^2 \int_{-1}^{1} (1 - t^2)^{1/2} R_n(t) dt
\]

\[
+ \alpha_n^2 \int_{-1}^{1} (1 - t^2)^{-1/2} R_n(t) dt,
\]

i.e.,

\[
(1 - \alpha_n^2) \int_{-1}^{1} (1 - t^2)^{-1/2} t^2 R_n(t) dt = \alpha_n^2 \int_{-1}^{1} (1 - t^2)^{1/2} R_n(t) dt.
\]

Since \(\int_{-1}^{1} R_n(t) dt = 1\) and \(\alpha_n = \sin(\pi/4n)\), it follows that
\[ \int_{-1}^{1} (1 - t^2)^{-1/2} t^2 R_n(t) \, dt \leq \tan^2 (\pi/4n) \leq 1/n^2 \]

and the lemma is proved in view of the inequality (3).

**Proof of the Theorem.** The operator \( K_n \) defined by (1) is clearly a positive linear operator. The inequality of Shisha and Mond mentioned earlier states that

\[ \| K_n[f] - f \| \leq (1 + \| K_n[1] \| ) \omega_f (\mu_n) + \| f \| \cdot \| K_n[1] - 1 \| \]

where \( \mu_n = \| K_n[(t-x)^2](x) \|^{1/2} \). Here, the operator \( K_n \) is applied to the variable \( t \in [-1/2, 1/2] \), while the sup norm \( \| \| \) is taken with respect to the variable \( x \in [-1/4, 1/4] \) (see [8, Theorem 1]). Hence, we have only to evaluate \( \mu_n, \| K_n[1] - 1 \| \) and \( \| K_n[1] \| \).

We have, first, for \( |x| \leq 1/4 \)

\[ K_n[(t-x)^2](x) = \int_{-1/2}^{1/2} (t-x)^2 R_n(t-x) \, dt \leq \int_{-1}^{1} t^2 R_n(t) \, dt \]

and so by the lemma

\[ \mu_n^2 \leq \int_{-1}^{1} t^2 R_n(t) \, dt \leq 1/n^2. \]

Next,

\[ 1 - K_n[1](x) = \int_{-1}^{1} R_n(t) \, dt - \int_{-1/2}^{1/2} R_n(t-x) \, dt \]

\[ = \int_{-x+1/2}^{1} R_n(t) \, dt + \int_{-1}^{1} R_n(t) \, dt. \]

Hence, for \( |x| \leq 1/4 \) we have

\[ |1 - K_n[1](x)| \leq \left( \int_{1/4}^{1} + \int_{-1}^{-1/4} \right) R_n(t) \, dt \]

\[ \leq 16 \left( \int_{1/4}^{1} + \int_{-1}^{-1/4} \right) t^2 R_n(t) \, dt \]

and so again by the lemma

\[ \| 1 - K_n[1] \| \leq 16 \int_{-1}^{1} t^2 R_n(t) \, dt \leq 16/n^2. \]
Finally, for $|x| \leq 1/4$

\[ (7) \quad K_1(x) \leq \int_{-1}^{1} R_n(t)dt = 1 \]

and (2) follows from (4), (5), (6) and (7).

REFERENCES


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