AN EXCEPTIONAL ARITHMETIC GROUP AND ITS EISENSTEIN SERIES

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1. **Introduction.** Let G_R be the simply-connected, real, Lie group of type E_7 which is isogenous to the full group of holomorphic automorphisms of a bounded symmetric domain in C^{27}. It is the purpose of this note to announce results on a certain arithmetic subgroup Γ of G_R and its automorphic forms; in particular, we have proved that the automorphic forms for Γ given by Eisenstein series have Fourier coefficients which are rational numbers with a certain Euler product expansion. Because the proofs are too long to give here, they will be presented elsewhere.

In this note, all our fields are of characteristic zero; we use $\boldsymbol{C}$, $\boldsymbol{R}$, $\boldsymbol{Q}$, and $\boldsymbol{Z}$ to denote respectively the complex numbers, the real numbers, the rational numbers, and the rational integers. If V is an algebraic group, algebra, or vector space defined over $\boldsymbol{Q}$, and if k is a field containing $\boldsymbol{Q}$, denote by V_k the group of k-rational points of V. It is not necessary that the family of all the fields we consider, ordered by inclusion, have a maximal element.

2. **Cayley numbers.** We denote by $\mathfrak{C}$ the ring of Cayley numbers constructed from the standard basis of eight units and multiplication table of [3]; this gives $\mathfrak{C}$ a $\boldsymbol{Q}$-structure, and $\mathfrak{C}_R$ is a division algebra. The ring $\mathfrak{C}_k$ has an involution $a \to \bar{a}$, from which we define the trace function $T: a \to a + \bar{a}$, a bilinear form $B: (a, b) \to T(a\bar{b})$, and norm $N: a \to a\bar{a}$. We identify k with the set of a in $\mathfrak{C}_k$ such that $a = \bar{a}$.

Coxeter [3] has constructed a subring of $\mathfrak{C}_R$, which we denote by $\mathfrak{o}$, which is a lattice, contained in $\mathfrak{C}_Q$, of the real vector space $\mathfrak{C}_R$, such that $\mathfrak{o} \cap \boldsymbol{R} = \boldsymbol{Z}$, and which has the further important properties: (1) $\mathfrak{o}$ is self-dual with respect to $B(\ ,\ )$; (2) if $a \in \mathfrak{o}$, then $T(a)$ and $N(a)$ are integers; (3) $\mathfrak{o}$ is maximal with respect to the preceding properties; and (4) if $\beta_1, \cdots, \beta_8$ is a basis of $\mathfrak{o}$ and if $a_1, \cdots, a_8$ are arbitrary elements of $\boldsymbol{Z}$, then there exists $a \in \mathfrak{o}$ such that $B(\beta_i, a) = a_i$, $i = 1, \cdots, 8$. We then have the

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LEMMA 1. *Let* $\mathfrak{o}_1 = \{a \in \mathfrak{o} \mid N(a) = 1\}$. *If* $x \in \mathfrak{C}_R$ *satisfies* $|B(x, c)| \leqq 1$ *for all* $c \in \mathfrak{o}_1$, *then* $N(x) \leqq \frac{1}{2}$.

The above facts about $\mathfrak{C}_R$ and $\mathfrak{o}$ are basic in proving the arithmetic part of our results.

3. **Exceptional Jordan algebras.** Let $\mathfrak{J}_k$ be the set of 3 by 3 matrices $X = (x_{ij})$ over $\mathfrak{C}_k$ such that $x_{ij} = \bar{x}_{ji}$; in particular, $\xi_i = x_{ii} \in k$. This becomes a Jordan algebra, supplied with the product $X \circ Y = \frac{1}{2}(XY + YX)$, where XY is the ordinary matrix product. One defines $\operatorname{tr}(X) = \sum_i x_{ii}$, an inner product (,) by $(X, Y) = \operatorname{tr}(X \circ Y)$, and a symmetric trilinear form (, ,) such that the associated cubic form $\det X$ has the formal appearance of a determinant [6]. Let $\times$ be the bilinear mapping of $\mathfrak{J}_k \times \mathfrak{J}_k$ into $\mathfrak{J}_k$ defined by $(X \times Y, Z) \equiv 3(X, Y, Z)$.

Let K be the cone of squares of elements of $\mathfrak{J}_R$, and let K^+ be its interior; if $X \in K^+$, then $\det X \neq 0$. We define $\mathfrak{J}_{\mathfrak{o}} = \{X \in \mathfrak{J}_R \mid X = (x_{ij}),\ x_{ij} \in \mathfrak{o}$ (in particular, $\xi_i \in \boldsymbol{Z})\}$. Then $\mathfrak{J}_{\mathfrak{o}}$ is a self-dual lattice with respect to (,).

4. **k-forms of E_6 and E_7.** Let $\mathcal{G}_k = \{g \in GL(\mathfrak{J}_k) \mid \det(gX) \equiv \det X\}$ and let $\mathcal{G}_{\mathfrak{o}} = \{g \in \mathcal{G}_Q \mid g\mathfrak{J}_{\mathfrak{o}} = \mathfrak{J}_{\mathfrak{o}}\}$. One defines [7] an automorphism $g \to g^*$ of $\mathcal{G}_k$ by $(gX, g^*Y) \equiv (X, Y)$. If $g \in \mathcal{G}_k$, then $g(X \times Y) = (g^*X) \times (g^*Y)$; moreover, $\mathcal{G}_{\mathfrak{o}}$ is stable under that automorphism. Using these facts, Lemma 1, and [6, Theorem 12], one may prove

PROPOSITION 1. *The group* $\mathcal{G}_R$ *is connected and* $\mathcal{G}_{\mathfrak{o}}$ *is a maximal discrete subgroup of* $\mathcal{G}_R$. *If two* $\boldsymbol{Q}$*-parabolic subgroups of* $\mathcal{G}_R$ *are conjugate, then they are conjugate by an element of* $\mathcal{G}_{\mathfrak{o}}$.

Let X and X' be two copies of $\mathfrak{J}_k$, and let Ξ and Ξ' be two copies of k. Define W_k to be the direct sum of vector spaces, $X \oplus \Xi \oplus X' \oplus \Xi'$. If $w \in W_k$, we may write w in terms of its components in the direct summands as $w = (X, \xi, X', \xi')$. Define [4] a quartic form J on W_k by $J(w) = (X \times X, X' \times X') - \xi \det X - \xi' \det X' - \frac{1}{4}((X, X') - \xi\xi')^2$ and a skew-symmetric bilinear form { , } on $W_k \times W_k$ by

$$\{w_1, w_2\} = (X_1, X_2') - (X_2, X_1') + \xi_1 \xi_2' - \xi_2 \xi_1',$$

where $w_i = (X_i, \xi_i, X_i', \xi_i')$. Define G_k to be the group of all linear transformations of W_k leaving J and { , } invariant. Let $W_{\mathfrak{o}}$ be the lattice of all $w \in W_Q$ such that $X, X' \in \mathfrak{J}_{\mathfrak{o}}$, and $\xi, \xi' \in \boldsymbol{Z}$, and define

$$\Gamma = G_{\mathfrak{o}} = \{g \in G_Q \mid gW_{\mathfrak{o}} = W_{\mathfrak{o}}\}.$$

PROPOSITION 2. *The group* G_R *is connected and* Γ *is a maximal dis-*

screte subgroup of G_R. If two Q-parabolic subgroups of G_R are conjugate, then they are conjugate by an element of Γ.

5. **The symmetric domain.** It is known [4] that G_R is that real, connected, and simply-connected form of E_7 which is isogenous to the holomorphic automorphism group of a noncompact symmetric, hermitian space D of 27 dimensions. The space D, which is complex analytically isomorphic to a bounded domain in $\mathbf{C}^{27}$, may also be identified with the tube domain [5]

$$\mathfrak{Z} = \{Z = X + iY \in \mathbf{C}^{27} \mid Y \in K^+\},$$

where K^+ is defined as in §3. The group G_R contains a subgroup P^+ which, via the action of G_R on $\mathfrak{Z}$, is isomorphic to the group of all translations: $Z \to Z + A$, $A \in \mathfrak{J}_R$; it also contains the inversion $\iota: Z \to -Z^{-1}$ (Jordan algebra inverse); and ι and the group P^+ generate G_R. Moreover, G_R contains a subgroup isomorphic to $\mathfrak{G}_R$; if $g \in \mathfrak{G}_R$, then the action of g on $\mathfrak{Z}$ is the complexification of its action on $\mathfrak{J}_R$. There is a unique parabolic subgroup $\mathcal{P}$ of G_R containing $\mathfrak{G}_R$ and P^+. Let $P_0^+ = P^+ \cap \Gamma$, identify $\mathfrak{G}_0$ with $\mathfrak{G}_R \cap \Gamma$, and let $\Gamma_0 = \Gamma \cap \mathcal{P}$.

6. **Automorphic forms.** If $g \in G_R$, denote by $j(Z, g)$ the functional (jacobian) determinant of g at Z (we view g as acting on the right). We remark that $j(Z, \iota) = (\det Z)^{-18}$, as follows from an easy calculation.

A holomorphic function f on $\mathfrak{Z}$ which satisfies the identity

$$j(Z, g)^m f(Z \cdot g) = f(Z), \qquad g \in \Gamma,\ Z \in \mathfrak{Z}, \tag{1}$$

where m is an even integer $\geqq 0$, is called an automorphic form of weight m for Γ. The relation (1) implies that f is invariant under Γ_0, and in particular $f(Z+A) = f(Z)$ for all $A \in \mathfrak{J}_0$. Hence f has a Fourier expansion

$$f(Z) = \textstyle\sum_T a(T)\epsilon((T, Z)),$$

where the sum is over $T \in \mathfrak{J}_0$ and $\epsilon(x) = e^{2\pi i x}$; by well-known principles [2, §10.14], one may even assert that $a(T) = 0$ if $T \notin K$.

7. **Eisenstein series.** If m is a positive even integer, we define

$$E_m(Z) = \textstyle\sum_{\gamma \in \Gamma/\Gamma_0} j(Z, \gamma)^m, \qquad Z \in \mathfrak{Z}.$$

This series converges uniformly on compact subsets of $\mathfrak{Z}$ [2, §7.2] and is an automorphic form of weight m for Γ. Hence it has a Fourier expansion

$$E_m(Z) = \textstyle\sum_{T \in \mathfrak{J}_0 \cap K} a_m(T)\epsilon((T, Z)). \tag{2}$$

One may prove, using in part some ideas of Siegel [8, §5], that the automorphic forms E_m generate the field of automorphic functions for Γ. It follows without great difficulty that if k is a field containing all the Fourier coefficients $a_m(T)$ for all m and T, then k is a field of definition for the Satake compactification [2, pp. 482–485] of $\mathfrak{Z}/\Gamma$, viewed as a projective variety.

8. **The Fourier coefficients.** One may prove

THEOREM. *All of the Fourier coefficients $a_m(T)$ are rational numbers.*

We indicate very briefly how this may be proved for the case $T\in K^+$ (so that $\det T\neq 0$). The general case is similar.

Denote by $\mathbf{Q}_p$ the p-adic completion of $\mathbf{Q}$, and by $\mathbf{Z}_p$, the ring of p-adic integers, for any rational prime p. Let $\mathfrak{J}_{0_p}=\mathfrak{J}_0\otimes_Z \mathbf{Z}_p$, and let $\mathfrak{g}_{0_p}$ be the stabilizer in $\mathfrak{g}_{Q_p}$ of $\mathfrak{J}_{0_p}$.

LEMMA 2. *If $u\in\mathfrak{J}_{Q_p}$, then there exists $g\in\mathfrak{g}_{0_p}$ such that $g\cdot u=(u'_{ij})$ satisfies $u'_{ij}=0$ if $i\neq j$.*

(This is true for every rational prime p, *including* 2.)

DEFINITION. *Let $u\in\mathfrak{J}_{Q_p}$. If $g\in\mathfrak{g}_{0_p}$ is as in Lemma 2, let $\kappa_p(u)$ denote the archimedean absolute value of the product of the reduced denominators of the nonzero elements of $g\cdot u$ (so that $\kappa_p(u)$ is a nonnegative power of p). Let $\kappa(u)=\prod_p \kappa_p(u)$, where the product is over all rational primes.*

Using the Poisson summation formula, one may prove, as in [8, §7], that if $Z\in\mathfrak{Z}$, then for any even positive m,

(3) $\sum_{\lambda\in\mathfrak{J}_0} \det (Z+\lambda)^{-18m} = \Delta(m)\cdot \sum_{T\in\mathfrak{J}_0\mathfrak{T}_0\cap K^+} \det T^{18m-9}\epsilon((T,Z))$,

where

$$\Delta(m) = 2^{54m}\cdot\pi^{54m-12}\cdot \prod_{n=0}^{2}\gamma(m-4n)^{-1},$$

γ being the gamma function. Using (3) to transform the series for E_m, one obtains, analogously to [8, §7],

$$a_m(T) = \Delta(m)\cdot\det T^{18m-9}\cdot \sum_{u\in\mathfrak{J}_Q \bmod \mathfrak{J}_0} \epsilon((T,u))\kappa(u)^{-18m}, \tag{4}$$

and the sum on the right is equal to $\prod_p S_p$, with

$$S_p = \sum_{u\in\mathfrak{J}_{Q_p} \bmod \mathfrak{J}_{0_p}} \epsilon_p((T,u))\kappa_p(u)^{-18m}, \tag{5}$$

where ϵ_p is a character on $\mathbf{Q}_p/\mathbf{Z}_p$ which is not trivial on $p^{-1}\mathbf{Z}_p$. Then, using a generalization of Hensel's lemma and some related ideas [1], one sees that for each p, the expression defining S_p is a sum of terms, each of which is itself a (finite) sum over an orbit of $\mathfrak{g}_{\mathfrak{o}_p}$ in $\mathfrak{J}_{Q_p}$ modulo $\mathfrak{J}_{\mathfrak{o}_p}$; those terms are zero except for a finite number of orbits of $\mathfrak{J}_{\mathfrak{o}_p}$, and for almost all p, one need only consider the sum over u such that $pu \in \mathfrak{J}_{\mathfrak{o}_p}$. From the first fact, one sees easily that each S_p is a rational number, and by the second fact, the calculation of S_p is reduced, for all but a finite number of p, to an enumerative calculation based on results of [7] and on elementary properties of character sums (in a way entirely different from the procedure in [8, §7]). The result is that for all but a finite number of p, we have

$$S_p = (1 - p^{-18m})(1 - p^{4-18m})(1 - p^{8-18m}). \tag{6}$$

Combining (4), (5), and (6), one obtains the result that $a_m(T)$ is a rational number. As we have said, our method of evaluating the series S_p is quite different from that of [8] for evaluating series analogous to S_p in a classical case. In [8], use is made of Gaussian sums and their relation to the interpretation of S_p as a representation density for quadratic forms. The absence, until now at least, of an analogous explanation in this exceptional case was what forced us to find another way of evaluating S_p. A problem of interest that remains is to find, if possible, an algebraico-geometric interpretation of the Euler factors (6), similar to the classical expression (in other cases) of such factors as representation densities of one quadratic form by another.

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