# RAMSEY'S THEOREM FOR $n$-DIMENSIONAL ARRAYS

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**Introduction.** An analogue to a theorem of Ramsey [5] has been conjectured for finite vector spaces by Gian-Carlo Rota. Namely, for each choice of positive integers $k$, $l$, $r$, and finite field $F=GF(q)$, there exists an integer $N(k, l, r; q)$ such that if $n \geqq N(k, l, r; q)$ and the $k$-dimensional subspaces of an $n$-dimensional vector space $V$ over $F$ are partitioned into $r$ classes, then some $l$-dimensional subspace of $V$ has all of its $k$-dimensional subspaces in one class. In this note we present a very general theorem of this type, a brief outline of its proof, and general applications, including some cases of Rota's Conjecture. Complete details will appear elsewhere.

**Notation.** Let $A=\{a_1, \cdots, a_t\}$ be a finite set with $t>1$ and let $H_p: A \to A$ be a permutation group on $A$. Define $H_c=\{\sigma_a: a \in A\}$ to be the set of maps of $A$ into $A$ given by $x^{\sigma_a}=a$ for all $x \in A$. $H$ will denote $H_c \cup H_p$. We can define an action of $H$ on $A^t$ by $(x_1, \cdots, x_t)^\sigma = (x_1^\sigma, \cdots, x_t^\sigma)$ for $x_i \in A$, $\sigma \in H$. Let $l_0$ denote $(a_1, \cdots, a_t) \in A^t$ and let $L_c=\{l_0^\sigma: \sigma \in H_c\}$, $L_p=\{l_0^\sigma: \sigma \in H_p\}$, $L=L_c \cup L_p$. We introduce the basic concept of a *k-parameter set.* For fixed nonnegative integers $k \leqq n$, let $\Pi=\{S_0, S_1, \cdots, S_k\}$ be a partition of the set $I_n = \{1, 2, \cdots, n\}$ with $S_i \neq \emptyset$ for $1 \leqq i \leqq k$. $S_0=\emptyset$ is possible. Let $f: I_n \to H$ have the property

$$f(i) \in H_c \text{ if } i \in S_0,$$
$$f(i) \in H_p \text{ otherwise.}$$

The set $P(\Pi, f)$ is defined by

$$P(\Pi, f) = \bigcup_{1 \leq i_0, i_1, \cdots, i_k \leq t} \{(x_1, \cdots, x_n); \quad x_j = a_{i_\nu}^{f(j)} \text{ if } j \in S_\nu\} \subseteq A^n.$$

Note that since $f(j) \in H_c$ for $j \in S_0$, $P(\Pi, f)$ consists of exactly $t^k$ elements of $A^n$.

DEFINITION. $P_k$ is *k-parameter set* of $A^n$ if and only if $P_k=P(\Pi, f)$ for some partition $\Pi$ and mapping $f$. Of course, we say that $P_k$ is a *k-parameter subset* of the $l$-parameter set $P_l \subseteq A^n$ if $P_k \subseteq P_l$ and $P_k$ is a $k$-parameter set of $A^n$.

**The main results.**

THEOREM 1. *For each choice of positive integers $k$, $l$, $r$ there exists an* *integer* $M(k, l, r)$ *such that if* $m \geqq M(k, l, r)$ *and the k-parameter subsets of an m-parameter set* $P_m \subseteq A^n$ *are partitioned into r classes, then there exists an l-parameter subset* $P_l \subseteq P_m$ *such that all k-parameter subsets of* $P_l$ *belong to the same class.*

Let us call a $k$-parameter set $P_k \subseteq A^n$ *normalized* if $f(j) = \sigma_{a_1}$ for all $j \in S_0$. We state the important

THEOREM 2. *The preceding theorem is valid if all parameter sets are required to be normalized.*

Before proceeding to the proof outline, we list several immediate corollaries to the theorems.

COROLLARY 1. *Given integers k and r, there exists an integer* $N(k, r)$ *such that if* $|A| \geqq N(k, r)$ *and the finite subsets of A are partitioned into r classes then there exist k disjoint nonempty subsets* $A_1, \cdots, A_k$ *of A such that all* $2^k - 1$ *unions* $\bigcup_{j \in J} A_j$, $\varnothing \neq J \subseteq \{1, 2, \cdots, k\} = I_k$, *are in the same class.*

This follows from Theorem 2, taking $A = \{0, 1\}$ and $H_p = \{e\}$.

COROLLARY 2 (J. FOLKMAN, J. SANDERS [6]). *Given integers k and r, there exists an integer* $N(k, r)$ *such that if* $n \geqq N(k, r)$ *and the set* $I_n$ *is partitioned into r classes, then there exist k integers* $a_1, \cdots, a_k$ *such that all sums* $\{ \sum_{i=1}^{k} \epsilon_i a_i : \epsilon_i = 0$ *or* $1$, *not all* $\epsilon_i = 0\}$ *are in the same class.*

This follows for Corollary 1 by interpreting the characteristic function of $A_i$ as the dyadic expansion of an integer $a_i$. For $k = 2$, Corollary 2 was first proved by Schur [7]. Schur's result can also be stated as follows:

Given $r$, there exists an integer $N(r)$ such that if $n \geqq N(r)$ and the set $I_n$ is partitioned into two classes, then the equation $x + y = z$ can be solved in one class. This is also a special case of

COROLLARY 3. *Let* $\mathfrak{L} = L_i(x_1, \cdots, x_m)$, $1 \leqq i \leqq n$ *be a system of homogeneous linear equations with the property that for each j,* $1 \leqq j \leqq m$, *there exists a solution* $(\epsilon_1, \epsilon_2, \cdots, \epsilon_m)$ *to the system* $\mathfrak{L}$ *with* $\epsilon_i = 0$ *or* $1$ *and* $\epsilon_j = 1$. *Then given an integer r there exists an integer* $N(r)$ *such that if* $n \geqq N(r)$ *and the set* $I_n$ *is partitioned into r classes, then* $\mathfrak{L}$ *can be solved in one class.*

This is similar to a result of R. Rado [3].

COROLLARY 4 (VAN DER WAERDEN [2]). *Given integers k and r there exists an integer* $N(k, r)$ *such that if* $n \geqq N(k, r)$ *and the set* $I_n$ *is partitioned into r classes, then at least one class contains an arithmetic progression of length k.*

This result is implied by the stronger

COROLLARY 5 (HALES-JEWETT [1]). *Let* $A=\{a_1, \cdots, a_t\}$ *be a finite set. Given an integer* $r$ *there exists an integer* $N(r, t)$ *such that if* $n \geqq N(r, t)$ *and the set* $A^n$ *is partitioned into* $r$ *classes, then there exists a set of* $t$ *elements of the form*

$$X_i = (x_{11}, \cdots, x_{1u}, a_i, x_{21}, \cdots, x_{2v}, a_i, \cdots, a_i, x_{d1}, \cdots, x_{dz}) \in A^n, \qquad 1 \leqq i \leqq t,$$

*all of which belong to one class.*

This follows from Theorem 1 by taking $A=\{a_1, \cdots, a_t\}$, $k=0$, $l=1$, $H_p=\{e\}$.

COROLLARY 6. *Given integers* $l$ *and* $r$ *and a finite field* $GF(q)$ *there exists an integer* $N(l, r, q)$ *such that if* $n \geqq N(l, r, q)$ *and the* 1*-dimensional subspaces of an* $n$*-dimensional vector space* $V$ *over* $GF(q)$ *are partitioned into* $r$ *classes, then* $V$ *contains an* $l$*-dimensional subspace* $V'$ *all of whose* 1*-dimensional subspaces are in one class.*

This follows from Theorem 2 by taking $A=GF(q)$, $H_p=$ mult. group of $GF(q)$, and $k=0$. The corresponding result for *affine* spaces over $GF(q)$ follows from Theorem 1. Corollary 6 was first proved for $q=2$ by D. Kleitman (unpublished) and $q=3$, 4 by B. L. Rothschild [4]. From the result for 1-dimensional *affine* subspaces, techniques of Rothschild [4] can be used to prove the result corresponding to Corollary 6 when 1*-dimensional* subspace is replaced by 2*-dimensional* subspace. It was conjectured by G.-C. Rota that Corollary 6 holds for $k$-dimensional subspaces in general.

Finally, as a more powerful application, let $C^n$ denote an $n$-dimensional cube in $E^n$. Let us say that a set $S_k$ of $2^k$ vertices of $C^n$ forms a *k-subspace* of $C^n$ if $S_k$ is contained in some $k$-dimensional euclidean subspace of $E^n$.

COROLLARY 7. *Given integers* $k$, $l$, $r$ *there exists an integer* $N(k, l, r)$ *such that if* $n \geqq N(k, l, r)$ *and the* $k$*-subspaces of* $C^n$ *are partitioned into* $r$ *classes, then there exists an* $l$*-subspace of* $C^n$ *all of whose* $k$*-subspaces are in one class.*

BRIEF OUTLINE OF PROOF OF THEOREM 1. Let $S(k; t_1, \cdots, t_r)$ denote the statement:

There exists an integer $M(k; t_1, \cdots, t_r)$ such that if $m \geqq M(k; t_1, \cdots, t_r)$ and the $k$-parameter subsets of an $m$-parameter set $P_m$ are partitioned into $r$ classes $C_1, C_2, \cdots, C_r$, then there exists an $t$, $1 \leqq t \leqq r$ and an $t$-parameter subset $P_{tt}$ of $P_m$ such that all the $k$-parameter subsets of $P_{tt}$ belong to class $C_t$.

We prove $S(k;\ t_1, \cdots, t_r)$ by multiple induction on $k$ and $t_1+t_2+\cdots+t_r$. We can assume $0 \leqq k$, $r \geqq 1$ and $t_i \geqq 1$ for all $i$. The first step in the induction is $S(0;\ t_1, \cdots, t_r)$. Once certain notational difficulties have been overcome, the proof of this statement is relatively straightforward. We assume $S(i;\ t_1, \cdots, t_r)$ has been established for $0 \leqq i < k$ and all $t_i$. Since $S(k;\ t_1, \cdots, t_r)$ is certainly valid if $t_1+t_2+\cdots+t_r \leqq rk$, we further assume that for some $t > rk$, $S(k;\ t_1, \cdots, t_r)$ is valid for all choices of $t_i$ with $t_1+\cdots+t_r < t$.

A critical step in the proof rests on the following fact. It is possible to define a map $M: L^n \rightarrow 2^{A^n}$ such that for each $l$-parameter set $P_l \subseteq A^n$ there exists an $(l-1)$-parameter set $P^*_{l-1} \subseteq L^n$ with $M(P^*_{l-1}) = P_l$ such that for "certain" $k$-parameter subsets $P_k \subseteq P_l$, there exists a $(k-1)$-parameter subset $P^*_{k-1} \subseteq P^*_{l-1}$ which makes the following diagram commutative:

$$\begin{array}{ccc} P^*_{k-1} & \subseteq & P^*_{l-1} \\ M \downarrow & \cdot & \downarrow M \\ P_k & \subseteq & P_l \end{array}$$

Thus, the original partition of the $k$-parameter sets $P_k$ into $r$ classes induces a partition of $(k-1)$-parameter sets $P^*_{k-1}$ to which we can apply the induction hypothesis. It turns out that the "remaining" $k$-parameter sets can be naturally embedded in a large parameter set to which we can again apply the preceding argument. After a large number of iterations of this procedure, we are left with a configuration of blocks of "remaining" $k$-parameter sets which in a certain sense is isomorphic to a large parameter set in which the blocks are identified with points. By then partitioning these point-blocks according to the way in which the corresponding constituent $k$-parameter subsets have been partitioned and applying $S(0;\ t_1', \cdots, t_r')$ for suitable $t_1', \cdots, t_r'$ we can extract a configuration of $k$-parameter sets from which the induction step can be completed fairly directly. Theorem 2 follows from Theorem 1 with little difficulty. As might be expected, the bounds provided on $M(k, l, r)$ by this proof are extremely large.

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# DECOMPOSITIONS OF $E^3$ INTO POINTS AND COUNTABLY MANY TREES

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In this paper $G$ always denotes a monotone decomposition of $E^3$, i.e., an upper semicontinuous decomposition into compact connected sets. $H_G$ denotes the set of nondegenerate elements of $G$, and $E^3/G$ denotes the quotient space of $E^3$ associated with $G$. "Homeomorphism" will mean "homeomorphism of $E^3$ onto itself." If $f$ is a homeomorphism, then $fG=\{f(g)\mid g\in G\}$ and $fG(X)=\bigcup\{f(g)\in fG\mid f(g)$ meets $X\}$. Also, $S(X, r)=\{p\in E^3\mid d(p, X)<r\}$, where $d$ is the usual metric.

The purpose of this paper is to outline some results leading to a proof of the following theorem. Details will be published elsewhere.

THEOREM 1. *If $H_G$ is countable and each element of $H_G$ is a tree consisting of tame arcs, then $E^3/G$ is topologically $E^3$.*

Recall that a tree is a space homeomorphic to a finite connected one-dimensional simplicial complex containing no simple closed curves. "Consisting of tame arcs" means that each arc of the tree corresponding to a one-simplex is tame. An example given by Fox and Artin [3, p. 987] shows that this condition on a tree is weaker than requiring the tree to be tame.

Theorem 1 extends a result of Bing [2, Theorem 3, p. 370] and answers a question posed by Armentrout [1, p. 5]. Theorem 2 of this paper is the main tool used in the proof of our main result. The methods used to prove Theorem 2 are analogous to those used by McAuley in [4, pp. 444–454].

DEFINITION. Let $b$ be a point of a compact set $B$ in $E^3$. $B$ is said to be shrinkable to near $b$ with respect to $G$ if given any open set $U$ containing $B\setminus\{b\}$ and any positive number $\epsilon$, there is a homeomor-