AVERAGING ITERATION IN A BANACH SPACE

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An infinite real matrix satisfying the Toeplitz conditions will be called regular; a regular matrix is *admissible* if it is nonnegative, lower triangular, and each row sums to 1.

Let T be a mapping of a Banach space X into itself. If $x \in X$ and A is regular, let $C(x,A,T)$ denote the sequence defined by $u_n = \sum_{k=1}^{\infty} a_{nk}T^{k-1}x$. If A is admissible, let $M(x, A, T)$ denote the pair of sequences given by $x_1 = x$, $v_n = \sum_{k=1}^{n} a_{nk}x_k$, $x_{n+1} = Tv_n$. The statement that $M(x, A, T)$ converges means that each of $\{x_n\}$ and $\{v_n\}$ converges and $\lim x_n = \lim v_n$. Since A is regular, the convergence of $\{x_n\}$ implies the convergence of $M(x, A, T)$.

For the identity matrix I, each sequence of $C(x, I, T)$ and $M(x, I, T)$ is just the ordinary sequence of iterates $\{T^{n-1}x\}$.

Since A is a regular matrix, $C(x, A, T)$ is regular, i.e., the convergence of $\{T^n x\}$, say to z, implies the convergence of $C(x, A, T)$ to z.

THEOREM 1. *If T is linear and A is admissible, then there is an admissible B such that $\{x_n\}$ of $M(x, A, T)$ is $\{u_n\}$ of $C(x, B, T)$. Hence $M(x, A, T)$ is regular for linear T.*

OUTLINE OF PROOF. To define B, first define, for each pair (j, u) of positive integers, $E_0(u) = a_{u1}$ and $E_j(u) = \sum_{k=2}^{u} a_{uk}E_{j-1}(k-1)$ (we use the convention that $\sum_{k=m}^{n} y_k = 0$ if $m > n$). Now let B be given by $b_{11} = 1$, $b_{m+1,1} = b_{1,n+1} = 0$, $b_{m+1,n+1} = E_{n-1}(m)$.

The proof follows easily once the following results are established.

(1) If $m \geqq n \geqq 1$ then $b_{m+1,n+1} = \sum_{j=1}^{m} a_{mj}b_{jn}$.

(2) If $n > m$ then $E_{n-1}(m) = 0$.

(3) If $m \geqq 2$ then $\sum_{j=1}^{m+1} b_{m+1,j} = \sum_{k=1}^{m} a_{mk} \sum_{j=1}^{k} b_{kj}$.

(4) If $r \geqq 2$ then $x_r = \sum_{k=2}^{r} E_{k-2}(r-1)T^{k-1}x$.

Among the theorems proved by Mann [4] when he introduced $M(x, A, T)$ were:

(a) If T is continuous and either sequence of $M(x, A, T)$ converges, then the other does, and their common limit is a fixed point of T.

(b) Let L be the admissible matrix whose nonzero entries in the nth row are all equal to $1/n$. If T is a continuous function from $[a, b]$ into itself with a unique fixed point p, then $M(x, L, T)$ converges to p, for any x in $[a, b]$.

It is easy to show that the analog of (b) for $C(x, L, T)$ does not hold.

Caldwell [1] has given the following example: Let E be the closed disc with radius 2 centered at 0 in the complex plane, suppose that $0<\phi<\pi/4$, and let F be the nonlinear function defined on E by $F(re^{i\theta})=(2r-r^2)e^{i(\theta+\phi)}$. For each nonzero x in E, $M(x, L, F)$ does not converge, but if $|x|=2$ then $\{F^n x\}$ converges to the unique fixed point 0. Hence $M(x, A, T)$ may not be regular if T is nonlinear. Further, it may be shown that for any x in E, $C(x, L, F)$ converges to 0.

To give a partial generalization of (b), we first define a *segmenting matrix* to be an admissible matrix A such that for each n, and for $k\leqq n$, $a_{n+1,k}=(1-a_{n+1,n+1})a_{nk}$. For such a matrix, v_{n+1} lies on the line segment joining v_n and $x_{n+1}=Tv_n$:

$$v_{n+1} = (1 - a_{n+1,n+1})v_n + a_{n+1,n+1}Tv_n.$$

L is a segmenting matrix.

THEOREM 2. *Let E be a convex compact subset of the complex plane, let T be a nonexpansive mapping of E into itself ($|Tx-Ty|\leqq|x-y|$ for all x and y in E) with a unique fixed point p, and let A be a segmenting matrix such that $\sum_{n=1}^{\infty}a_{nn}(1-a_{nn})$ diverges. If $x\in E$ then $M(x, A, T)$ converges to p.*

OUTLINE OF PROOF. It is not difficult to modify the problem so that $p=0$. Then $\{|v_n|\}$ is nonincreasing; suppose that $b=\lim |v_n|>0$.

Since E is compact and $\{v_n\}$ does not converge to 0, 0 is not a cluster value of $\{v_n-x_{n+1}\}$. For each n, $v_n\neq x_{n+1}$. Thus there is a d such that $0<d<b$ and $|v_n-x_{n+1}|\geqq d$ for each n.

Using the fact that for any three complex numbers x, y, and z, if $x\neq 0, z\neq 0$, $|x-y|=|y-z|$, and if t is in $[0, 1]$, then $|tx+(1-t)z-y|^2 = |y-z|^2-t(1-t)|z-x|^2$, we show that for each n, $|v_{n+1}|^2 \leqq |v_n|^2-a_{n+1,n+1}(1-a_{n+1,n+1})d^2$. Hence, by induction, for each n,

$$|v_{n+1}|^2 \leqq |v_1|^2 - d^2 \sum_{k=2}^{n+1} a_{kk}(1 - a_{kk}).$$

This yields a contradiction since $\sum_{n=1}^{\infty} a_{nn}(1-a_{nn})$ diverges.

Except for Theorem 6 below, suppose that T is nonexpansive and that X is uniformly convex. In this setting, G. Birkhoff's mean ergodic theorem says that if T is linear, then for each x, $C(x, L, T)$ (that is, the sequence $\{(1/n)\sum_{k=1}^{n} T^{k-1}x\}$) converges to a fixed point of T. Let P denote the segmenting matrix such that $p_{n+1,n+1}=\frac{1}{2}$ for each n.

CONJECTURE. If T is linear then $M(x, P, T)$ converges. (Here,

$$v_{n+1} = 1/2^n \sum_{k=1}^{n+1} \binom{n}{k-1} T^{k-1}x.)$$

THEOREM 3. *The conjecture holds if X is finite dimensional.*

This theorem is really a corollary of the following results, which do not require finite dimensionality. The first lemma may be obtained as a corollary of a result of Browder and Petryshyn [2].

LEMMA 4-1. *For the process $M(x, P, T)$, if T is linear, then $\{v_n - x_{n+1}\}$ has limit* 0.

LEMMA 4-2. *If T is linear and if $\{v_n\}$ has a cluster value, then $M(x, P, T)$ converges.*

THEOREM 4. *If T is linear and demicompact ($\{u_n\}$ bounded and $\{u_n - Tu_n\}$ convergent imply that $\{u_n\}$ has a convergent subsequence), then $M(x, P, T)$ converges.*

COROLLARY. *If T is linear and compact then $M(x, P, T)$ converges.*

If $0 < \lambda < 1$ and $f \in X$, let $V_\lambda = \lambda I + (1-\lambda)(T+f)$.
We obtain corollaries for the iteration process $\{V_\lambda^n x\}$ of theorems given by Browder and Petryshyn [2], [3].

THEOREM 5. *If $f \in X$ then a solution of $u = Tu + f$ exists if and only if, for each x, $\{V_\lambda^n x\}$ is bounded.*

THEOREM 6. *If T is a bounded linear mapping of a Banach space into itself which is asymptotically convergent (for each x, $\{T^n x\}$ converges) and if f is in the range of $I - T$, then $\{V_\lambda^n x\}$ converges to a solution of $u = Tu + f$.*

There are elementary examples of bounded linear mappings which are not asymptotically convergent and for which $\{V_\lambda^n x\}$ converges, but the process given by $\phi_0 = x$, $\phi_{n+1} = T\phi_n + f$ does not converge.

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