MAPPING CYLINDERS AND THE ANNULUS CONJECTURE

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Suppose \( f \) is an embedding of the \( n \)-sphere \( S^n \) into the \((n+1)\)-sphere \( S^{n+1} \); \( f \) is said to be locally flat at \( x \in S^n \) if there is a neighborhood \( U \) of \( f(x) \) in \( S^{n+1} \) such that the pair \((U, U \cap f(S^n))\) is homeomorphic to \((E^{n+1}, E^n)\) where \( E^i \) is Euclidean \( i \)-space; i.e., there exists a homeomorphism \( h: U \to E^{n+1} \) such that \( h(U \cap f(S^n)) = E^n \cong E^n \times 0 \subseteq E^n \times E^1 = E^{n+1} \). Brown \([2], [3]\) has shown that if \( f \) is locally flat at each point of \( S^n \), then the closure of each complementary domain of \( f(S^n) \) in \( S^{n+1} \) is homeomorphic to an \((n+1)\)-cell. One of the outstanding unsolved problems in topology of manifolds is the annulus conjecture.

Suppose \( f, g \) are two locally flat embeddings (i.e., \( f \) and \( g \) are locally flat at each point of \( S^n \)) of \( S^n \) into \( S^{n+1} \) such that \( f(S^n) \cap g(S^n) = \emptyset \). The connected submanifold \( A^{n+1} \) of \( S^{n+1} \) whose boundary is \( f(S^n) \cup g(S^n) \) is called a pseudo-annulus. The annulus conjecture is that \( A^{n+1} \) is homeomorphic to \( S^n \times [0, 1] \). If \( f, g \) are both either piecewise linear or differentiable maps or if \( n \leq 2 \), then the conjecture is true.

This paper was motivated by an attempt to construct a counterexample to the annulus conjecture. Let \( \rho: S^n \to S^n \) be a continuous map. The mapping cylinder of \( \rho \), \( \text{Map}(\rho) \), is the decomposition space formed from the disjoint union \((S^n \times [0, 1]) \cup S^n \) by identifying \((x, 1)\) with \( \rho(x) \) for each \( x \in S^n \). The idea was to find a map \( \rho: S^n \to S^n \) such that \( \text{Map}(\rho) \) is an \((n+1)\)-manifold which is not homeomorphic to \( S^n \times I \); for example, one might attempt to construct such a \( \rho \) by using a variation of Bing’s example \([1]\) of an upper semicontinuous decomposition of \( S^3 \) which yields \( S^3 \) but some of whose nondegenerate elements are spheres. By Proposition 2, \( \text{Map}(\rho) \) would be a pseudo-annulus and hence a counterexample. However, we show that this is impossible in dimension 3; i.e., if \( \text{Map}(\rho) \) is a manifold, then it is homeomorphic to \( S^n \times I \).

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Let \( \rho: S^n \to S^n \) be a continuous map such that \( \text{Map}(\rho) \) is an \((n+1)\)-manifold. Let \( \pi: (S^n \times I) \cup S^n \to \text{Map}(\rho) \) be the natural projection.

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PROPOSITION 1. The boundary of \( \text{Map} \ (p) \) is the union of the two \( n \)-spheres \( (S^n \times 0) \) and \( (S^n \times 1) \).

PROOF. Suppose \( M \) has one boundary component. Note that \( M \) is homotopically equivalent to \( S^n \). In the exact sequence

\[
H_n(\partial M) \xrightarrow{i_*} H_n(M) \rightarrow H_n(M, \partial M)
\]

\( i_* \) is the zero map and by Poincaré Duality, \( H_n(M, \partial M) \) is isomorphic to \( H^1(M) = 0 \). Hence \( H_n(M) = 0 \), a contradiction.

PROPOSITION 2. \( \text{Map} \ (p) \) is a pseudo-annulus.

PROOF. By attaching an \( (n+1) \)-cell to each boundary component of \( M \), one obtains a closed manifold \( S \). It is easy to see that \( S \) is the union of two open \( (n+1) \)-cells and hence by [2], \( S \) is an \( (n+1) \)-sphere in which \( M \) appears as a pseudo-annulus.

PROPOSITION 3. If \( n \neq 4 \), then \( p \) is a cellular map; i.e., if \( x \in S^n \), then \( p^{-1}(x) = \bigcap_{i=1}^{n+1} C_i \) where \( C_i (\subseteq \text{interior } C_{i-1}) \) are closed \( n \)-cells in \( S^n \).

PROOF. Let \( U \) be a contractible open subset of \( S^n \). Define \( g: p^{-1}U \rightarrow U \) by \( g = \text{proj} \mid p^{-1}U \). Map \( (g) \) is a contractible open subset of Map \( (p) \) for Map \( (g) = r^{-1}U \) where \( r \) is the canonical deformation retraction of Map \( (p) \) onto image \( (p) = S^n \). Thus Map \( (g) \) is an \( (n+1) \)-manifold. Since \( U \) is collared in Map \( (g) \) [3], Map \( (g) - U \) is contractible. But Map \( (g) - U \) deformation retracts to \( p^{-1}U \) and hence \( p^{-1}U \) is contractible. By Lacher [5, Theorem 2] for any open subset \( V \) of \( S^n \), \( p^{-1}(V) \rightarrow V \) is a proper homotopy equivalence. Let \( x \in S^n \), then \( x = \bigcap_{i=1}^{n+1} D_i \) where \( D_i (\subseteq \text{interior } D_{i-1}) \) are closed \( n \)-cells in \( S^n \). Since \( p^{-1}(x) = \bigcap_{i=1}^{n+1} p^{-1}D_i \), if we want to show that \( p^{-1}(x) \) is cellular, it is sufficient to show that there exists an \( n \)-cell \( C_i \) in \( p^{-1} (\text{interior } D_i) \) for each \( i \) such that \( p^{-1}(x) \) is contained in the interior of \( C_i \). From above \( p: p^{-1} (\text{interior } D_i) \rightarrow \text{interior } D_i \) is a proper homotopy equivalence; since interior \( D_i \) is 1-connected at infinity, \( p^{-1} (\text{interior } D_i) \) is 1-connected at infinity. For \( n = 3 \), \( p^{-1} (\text{interior } D_i) \) is an open 3-cell by Edwards [4]. For \( n \geq 5 \), we apply Stallings [7]. It is now easy to find \( C_i \).

THEOREM. If \( p: S^3 \rightarrow S^3 \) is a continuous map and Map \( (p) \) is a manifold, then Map \( (p) \) is homeomorphic to \( S^3 \times I \).

PROOF. By Proposition 3, \( p \) is a cellular map. By Price [6] there exists a pseudo-isotopy \( H: S^3 \times I \rightarrow S^3 \times I \) (i.e., \( H \) is level preserving and the map \( H_t: S^3 \rightarrow S^3 \), defined by \( H(x, t) = (H_t(x), t) \), is a homeomorphism for \( t \in [0, 1) \)) such that \( H_0 \) is the identity map and \( H_1 = p \).
Define $\phi: \text{Map}(\mathcal{P}) \to S^3 \times I$ by $H\pi^{-1}(x)$. It is easily seen that $\phi$ is a homeomorphism using the fact that $\pi$ is an open map.

REFERENCES


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