SMOOTH HOMOTOPY PROJECTIVE SPACES
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Introduction. In [5] we considered certain fixed point free involutions on Brieskorn manifolds as weakly complex bordism elements. In [4] we considered associated examples of smooth normal invariants for real projective spaces, settling the realizability question for dimensions \( \neq 1 \mod 4 \) and the desuspendability question for dimensions \( 4k+1 \). The object of this study is the classification of these smooth normal invariants given by the Brieskorn examples. Our results overlap somewhat with Atiyah and Bott [2] as well as Browder [3], but our methods are entirely different and our results rather more refined. Full details of these and related results will appear elsewhere.

1. Smooth normal invariants. Following Sullivan [6], we regard a smooth normal invariant of a space \( X \) as an element of \( [X, G/O] \). Of course, we have \( G/O \cong SG/SO \). We need the fibers \( SG/Spin \) of \( BSpin \rightarrow BSG \) and \( SO/Spin \cong P^\infty \) of \( BSpin \rightarrow BSO \). The spaces \( SG/SO \), \( SG/Spin \), \( SO/Spin \) have their Whitney \( H \)-space structures under which the sequence

\[ SO/Spin \rightarrow SG/Spin \rightarrow SG/SO \]

is a multiplicative fibration.

A map \( \mu : SG/Spin \rightarrow BO \) is constructed as follows. Let \( \gamma_n \) denote the universal fiber space over \( BSG_n \) with fiber \( S^{n-1} \), \( \beta_n \) the pullback to \( BSpin_n \), and \( \alpha_n \) the pullback to \( SG_n/Spin_n \); also, let \( \epsilon_n \) denote the \( S^{n-1} \) fibration over a point. Corresponding to the commutative diagram

\[ \begin{array}{ccc}
SG_n/Spin_n & \xrightarrow{\beta_n} & BSpin_n \\
\{pt\} & \xrightarrow{\gamma_n} & BSG_n \\
\end{array} \]

of spaces, there is the commutative diagram

\[ \begin{array}{ccc}
\alpha_n & \xrightarrow{\beta_n} & \gamma_n \\
\epsilon_n & \xrightarrow{} & \gamma_n \\
\end{array} \]

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of induced $S^{n-1}$ fibrations. Passing to Thom spaces we obtain a commutative diagram of spectra

\[
\begin{array}{ccc}
N(SG/\text{Spin}) & \xrightarrow{MSpin} & MSG \\
\downarrow S & & \downarrow \text{MSpin} \\
\end{array}
\]

where $S$ is the sphere spectrum and $N(SG/\text{Spin})$ is the spectrum with $N(SG/\text{Spin})_n = T(\alpha_n) \simeq S^n \wedge (SG_n/\text{Spin}_n)^+$. There is a map of spectra $MSpin \to bO$, where $bO$ is the $\Omega$-spectrum with $bO_0 = \mathbb{Z} \times BO$, defining the $kO$-orientation of Spin cobordism. Now the composition

\[N(SG/\text{Spin}) \to MSpin \to bO\]

of maps of ring spectra defines $\mu: SG/\text{Spin} \to BO$ in the usual way, since $N(SG/\text{Spin})$ and $S \wedge (SG/\text{Spin})^+$ are equivalent.

(1.1) **Theorem.** The map $\mu: SG/\text{Spin} \to BO$ is an $H$-map from the Whitney structure of $SG/\text{Spin}$ to the tensor product structure of $BO$.

The following is a central fact in our study.

(1.2) **Theorem.** The composition

\[P^\infty \simeq SO/\text{Spin} \to SG/\text{Spin} \to BO\]

classifies the canonical line bundle $\eta$ over $P^\infty$.

Since this map $\eta: P^\infty \to BO$ splits the map $w_1: BO \to K(\mathbb{Z}_2, 1) \simeq P^\infty$, we get the following easily.

(1.3) **Corollary.** The above maps fit into a commutative diagram

\[
\begin{array}{ccc}
P^\infty & \xrightarrow{\eta} & BO \\
\downarrow \mu & & \downarrow \nu \\
P^\infty & \xrightarrow{\mu} & BSO
\end{array}
\]

of maps of $H$-spaces ($BO$ and $BSO$ with the tensor product structure), each row being a multiplicative fibration.

Using Poincaré duality for Spin cobordism theory, we are able to make explicit computations of the $kO$-orientation $\mu: SG/\text{Spin} \to BO$. This amounts to interpreting $N(SG/\text{Spin}) \to MSpin$ in terms of Spin bordism of Spin manifolds.

For a finite $CW$ complex $X$, we take a homotopy equivalent compact Spin manifold $M^n$. An element of $[M^n, SG/\text{Spin}]$ is represented by a spherical fiber bundle over $M^n$ with Spin structural group and an $SG$ (i.e. degree $+1$ fiber homotopy) trivialization. Using transverse
regularity on the $SG$ trivialization, we obtain an element $[V^m, \partial V^m; e] \in \Omega^m_{\text{Spin}}(M^m, \partial M^m)$ of degree $+1$ ($e$ is the bundle projection). The Poincaré duality isomorphism sends this to an element of $\Omega^0_{\text{Spin}}(M^m)$ with augmentation $+1$. Now applying the $kO$-orientation of Spin cobordism, we obtain a virtual line bundle over $M^m$, i.e. an element of $[M^m, BO]$. This describes the natural map $[X, \mu]: [X, SG/\text{Spin}] \to [X, BO]$.

2. Spin cobordism of projective spaces. At this point the Brieskorn examples are brought into play. The degree $q$ maps $h: Q^q_{4k-1} \to P^{4k-1}$ of [5, §3] may be taken to represent elements $[Q^q_{4k-1}, h] \in \Omega^q_{\text{Spin}}(P^{4k-1})$ in a canonical way so that the following analogue of [5, Theorem 3.4] holds.

(2.1) **Theorem.** The Poincaré duals $b^q_{4k-1} \in \Omega^q_{\text{Spin}}(P^{4k-1})$ of the elements $[Q^q_{4k-1}, h] \in \Omega^q_{\text{Spin}}(P^{4k-1})$ satisfy $b^q_{4k-1} = q \cdot 1$.

It happens that $h: Q^q_{4k-1} \to P^{4k-1}$ is a diffeomorphism, making it clear what Spin structures to use. Actually, the above holds equally well for $Sp$ and $SU$ cobordism of $P^{4k-1}$.

To see what happens for other dimensions, just cut the maps $h: Q^q_{4k-1} \to P^{4k-1}$ down to a suitably small smooth regular neighborhood of $P^n \subset P^{4k-1}$, obtaining $h: V^q_{4k-1,n} \to M^{4k-1,n}$, say. Note that the Brieskorn examples $Q^n_q$ of [4] are the transverse regular inverse images in $Q^q_{4k-1}$ of $P^n \subset P^{4k-1}$. The resulting elements $b^n_q \in \Omega^q_{\text{Spin}}(P^n)$ are just the inclusion induced pullbacks of the elements $b^q_{4k-1}$ for $n \leq 4k - 1$.

(2.2) **Corollary.** In $\Omega^q_{\text{Spin}}(P^n)$ we have $b^n_q = q \cdot 1$.

The double covering $e: S^{4k-1} \to P^{4k-1}$ defines the element $[S^{4k-1}, e] \in \Omega^q_{\text{Spin}}(P^{4k-1})$ of degree $+2$. For each $n$, Poincaré duality and pulling back from $P^{4k-1}$ to $P^n$ produce the elements $s^n_q \in \Omega^q_{\text{Spin}}(P^n)$ with augmentation $+2$. To emphasize the fact that $s^n \neq 2 \cdot 1$ we point out the following.

(2.3) **Theorem.** Under the $kO$-orientation morphism

$$\Omega^*_{\text{Spin}}(P^n) \to kO^*(P^n)$$

$$s^n \to 1 + \eta,$$ where $\eta$ is the canonical line bundle.

The following, on the other hand, is straightforward geometry.

(2.4) **Proposition.** For any integer $j$ there is a map

$$h': (V^q_{4k-1,n}, \partial V^q_{4k-1,n}) \to (M^{4k-1,n}, \partial M^{4k-1,n})$$
of degree \( q + 2j \) such that the Poincaré dual of \([V_\Theta^{q-1,n}, \partial V_\Theta^{q-1,n}; h']\) is 
\( b^*_\Theta + j \cdot s^n \in \Omega_{\text{Spin}}^0(P^n)\).

3. The Brieskorn normal invariants. In [4] we pointed out how the Brieskorn examples \( Q_{4d+1}^n \) produce smooth normal invariants of \( P^n \). Here we show how these examples lead to elements of \([P^n, SG/\text{Spin}]\) compatible with the machinery of the previous section.

As remarked in [4], each \( Q_{4d+1}^n \) is homotopy equivalent to \( P^{4k+1} \). Consequently the map 
\[
P_{2d+1}^n: (V_{2d+1}^{4k+3, 4k+1}, \partial V_{2d+1}^{4k+3, 4k+1}) \to (M^{4k+3, 4k+1}, \partial M^{4k+3, 4k+1})
\]
of degree +1 provided by (2.4) is a homotopy equivalence of Spin manifolds. Now the construction of [6] produces in our case a “classifying” \( SG/\text{Spin}\)-bundle over \( M^{4k+3, 4k+1} \to P^{4k+1} \) for \( h_{2d+1} \), i.e. an element \( a_{2d+1}^n \in [P^{4k+1}, SG/\text{Spin}] \). Restriction to \( P^n \subset P^{4k+1} \) defines the element \( a_{2d+1}^n \in [P^n, SG/\text{Spin}] \).

Now we follow the description of \([P^n, \mu]\) as given in §1. We find that \( a_{2d+1}^n \) leads to the element \( b_{2d+1}^n - d \cdot s^n \in \Omega_{\text{Spin}}^0(P^n) \) which by (2.2) and (2.3) is sent (under \( kO\)-orientation) to the element \( 1 + d \cdot \xi \in kO^0(P^n) \), where \( \xi = 1 - \eta \). This gives our main result on the Brieskorn examples.

(3.1) Theorem. Under \( kO\)-orientation we have

\[
[P^n, \mu](a_{2d+1}^n) = 1 + d \cdot \xi
\]
as a virtual line bundle over \( P^n \).

Numerous results follow from (3.1), in particular the following which stems from (1.3).

(3.2) Theorem. The natural maps

\[ [P^n, \mu]: [P^n, SG/\text{Spin}] \to [P^n, BO], \]
\[ [P^n, \nu]: [P^n, G/O] \to [P^n, BSO], \]

are epimorphisms of groups (where \( BO \) and \( BSO \) have their tensor product structures).

Results of Browder [3] can be applied to show that the epimorphisms in (3.2) are canonically split—in fact, that \([P^n, \nu]\) classifies the Brieskorn examples of smooth normal invariants for \( P^n \). Similarly, \([P^n, \mu]\) classifies the Brieskorn elements \( a_{2d+1}^n \in [P^n, SG/\text{Spin}] \). Browder’s results alone do not give this except for \( n \leq 5 \).

4. Numerical results. By results of Adams [1], \( kO^0(P^n) = KO(P^n) \)
is the commutative ring generated by 1 and $\xi = 1 - \eta$ subject to the relations $\xi^2 = 2 \cdot \xi$ and $a_{n+1} \cdot \xi = 0$, where $a_k$ is given by the table:

<table>
<thead>
<tr>
<th>$k$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>$\ldots$</th>
<th>$j+8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_k$</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>4</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>$\ldots$</td>
<td>$16a_j$</td>
</tr>
</tbody>
</table>

Now it is clear that the multiplicative group of virtual line bundles $1 + 4 \cdot \xi \in [P^n, BO]$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}_{a_{n+1}/2}$. Moreover, the two generators $1 - \xi = \eta$ and $1 - 2 \cdot \xi = 2 \cdot \eta - 1$ of $[P^n, BO]$ generate the $\mathbb{Z}$ and $\mathbb{Z}_{a_{n+1}/2}$ factors corresponding to $[P^n, P^n]$ and $[P^n, BSO]$, respectively—as indicated in the following diagram:

$\begin{array}{c}
[P^n, P^n] \cong [P^n, BO] \cong [P^n, BSO] \\
\downarrow \downarrow \downarrow \\
\mathbb{Z} \cong \mathbb{Z} \times \mathbb{Z}_{a_{n+1}/2} \cong \mathbb{Z}_{a_{n+1}/2}
\end{array}$

(4.1) **Theorem.** There are $a_{n+1}/2$ distinct Brieskorn smooth normal invariants of $P^n$.

(4.2) **Corollary.** There are $a_{4k+3}/2 = 2^{2k}$ smoothly distinct Brieskorn homotopy projective $(4k+1)$-spaces. For $k > 0$, these yield only 4 combinatorially distinct homotopy projective $(4k+1)$-spaces.

(4.3) **Corollary.** For $n \neq 1 \mod 4$ and $n > 5$, there are $a_{n+1}/4$ smoothly distinct homotopy projective $n$-spaces which yield only 2 combinatorially distinct homotopy projective $n$-spaces.

**References**


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