We state here sufficient conditions for certain minimal surfaces to be differentiable at boundary points.

Let $m$ and $n$ be integers with $1 < m < n$. We adopt the notation of [3]. See also [2]. In particular, $I_m(\mathbb{R}^n)$ is the group of $m$ dimensional integral currents in $\mathbb{R}^n$. If $T \in I_m(\mathbb{R}^n)$, $M(T)$ is the mass of $T$ and $\partial T$ is the boundary of $T$; if $a \in \mathbb{R}^n$, $\Theta^m(\|T\|, a)$ is the $m$ dimensional density of the variation measure $\|T\|$ at $a$.

If $T \in I_m(\mathbb{R}^n)$, we say $T$ is minimal if there exists $r > 0$ such that $M(T) \leq M(S + T)$ whenever $a \in \mathbb{R}^n$, $S \in I_m(\mathbb{R}^n)$, $\partial S = 0$ and $\text{spt} \, S \subset \{x: \|x - a\| < r\}$. Given $B \in I_{m-1}(\mathbb{R}^n)$ with $\partial B = 0$, it is shown in [3] that there exists $T \in I_m(\mathbb{R}^n)$ such that $\partial T = B$ and $M(T) \leq M(S + T)$ whenever $S \in I_m(\mathbb{R}^n)$ with $\partial S = 0$.

**Theorem.** Suppose $T \in I_m(\mathbb{R}^n)$, $T$ is minimal, $a \in \text{spt} \, \partial T$, $\rho \geq 2$, $\Theta^{m-1}(\|\partial T\|, a) = 1$ and $\text{spt} \, \partial T$ intersects some neighborhood of $a$ in a class $\rho$ (real analytic) $m-1$ dimensional submanifold of $\mathbb{R}^n$.

1. If $\Theta^m(\|T\|, a) = 1/2$, then the intersection of $\text{spt} \, T$ with some neighborhood of $a$ is a subset of some class $\rho-1$ (real analytic) $m$ dimensional submanifold of $\mathbb{R}^n$.

2. If there exist independent linear functionals $\alpha_i$, $i = 1, \ldots, n - m + 1$, on $\mathbb{R}^n$ such that either

\[
\text{spt} \, \partial T \subset \{x: \alpha_i(x - a) \geq 0, i = 1, \ldots, n - m + 1\},
\]

or there is $r > 0$ such that

\[
\{x: \|x - a\| < r\} \cap \text{spt} \, T \subset \{x: \alpha_i(x - a) \geq 0, i = 1, \ldots, n - m + 1\},
\]

then $\Theta^m(\|T\|, a) = 1/2$.

**Corollary.** Suppose $\rho \geq 2$ and $B$ is the $m-1$ dimensional integral current corresponding to some compact oriented class $\rho$ (real analytic) $m-1$ dimensional submanifold $N$ of $\mathbb{R}^n$. If $N$ lies on the boundary of some uniformly convex open subset of $\mathbb{R}^n$ and $T \in I_m(\mathbb{R}^n)$ is minimal

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with \( \partial T = B \) then there is \( r > 0 \) and a class \( p - 1 \) (real analytic) \( m \) dimensional submanifold \( M \) of \( \mathbb{R}^n \) such that

\[
\text{spt } T \cap \{ x : \text{distance } (x, N) < r \} \subset M.
\]

Statement (1) is proved by combining the interior regularity results of [1] or of [8] with the construction of certain surfaces of dimension \( n - 1 \) which are barriers for the \( m \) dimensional area problem, and then applying the higher differentiability results of [6] and [7]. Statement (2) is proved by applying a variational argument to a tangent cone of \( T \) at \( a \). The corollary is an elementary consequence of the theorem.

These results remain true if we replace the group \( I_m(\mathbb{R}^n) \) by the group of flat chains over the integers modulo 2, as in [5]. If \( m = 2 \) and \( n = 3 \), the only boundary density that occurs on the smooth boundary of a minimal chain is one half. In view of the interior regularity results of [4], a minimal flat chain over the integers modulo 2 in \( \mathbb{R}^3 \) which spans a finite family of smooth curves must be free from singularities of any kind.

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REFERENCES


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