LEES' IMMERSION THEOREM AND THE TRIANGULATION OF MANIFOLDS

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In [4] Lees proves the following immersion theorem for topological manifolds: Let \( M, M', Q \) be topological manifolds, \( M \) a compact locally flat submanifold of the open manifold \( M' \), with \( \dim M' = \dim Q = q \), and \( \partial M = \emptyset \). Write \( \text{Im}_{M'}(M, Q) \) for the s.s. complex of \( M' \) immersions of \( M \) in \( Q \); and write \( R(TM'/M, TQ) \) for the s.s. complex of representative germs of \( TM'/M \) in \( TQ \). A representative germ is a bundle map of the tangent bundle \( TM' \) of \( M' \), restricted to a neighborhood of \( M \), into the tangent bundle \( TQ \) of \( Q \). Two germs are identified if they agree over a common neighborhood of \( M \).

**Theorem (Lees).** If \( M \) has a handle decomposition with all handles of index \( < Q \); the differential \( d: \text{Im}_{M'}(M, Q) \to R(TM'/M, TQ) \) is a homotopy equivalence.

We show here how to simplify some of the hypotheses of this theorem and give applications to the problem of triangulating topological manifolds.

**Theorem A.** In the following two cases, the assumption that \( M \) has a handle decomposition may be dropped in Lees' Immersion Theorem.

1. \( \dim M < \dim Q \).
2. \( \dim M = \dim Q \geq 5 \), and \( Q \) is a piecewise linear (PL) manifold.

Of course, if \( M \) is a PL-manifold, \( M \) has a handle decomposition, and hence Lees' theorem applies.

**Theorem B.** In the following cases, \( R(TM'/M, TQ) \) may be taken to be the s.s. complex of ordinary bundle maps of \( TM' \), restricted to \( M \), into \( TQ \).

1. \( \dim M = \dim Q \).
2. \( \dim M < \dim Q \), \( M \) a closed submanifold of \( M' \) and \( M \) the homotopy type of a locally finite simplicial complex.

We will say that an \( R^k \)-bundle \( \xi \) over a space dominated by a locally finite simplicial complex \( K \) admits a PL-bundle structure, if the pullback of \( \xi \) over \( K \) is the underlying topological bundle of a PL-\( R^k \)-bundle.

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bundle over $K$. (This is independent of choice of dominating complex and maps.)

Now let $M$ be closed, $\dim M = n \geq 5$, and $M$ simply connected. Write $M^0 = M$-open $n$-ball.

**Theorem C.** If the tangent bundle of $M^0$ admits a PL-bundle structure, $M$ admits a PL-manifold structure.

Finally we have

**Theorem D.** Let $M_1$, $M_2$ be closed PL-manifolds; $\dim M_i \geq 5$, and $M_i$ simply connected, $i = 1, 2$. A homeomorphism $h: M_1 \to M_2$ is concordant (or weakly isotopic) to a PL-homeomorphism, if and only if the topological bundle map $dh \oplus 1: T(M_1) \oplus 1 \to T(M_2) \oplus 1$ is homotopic through topological bundle maps to a PL-bundle map.

**Proof of A.** Since the essential trick in proving (1) is also used in proving (2), we only prove the latter.

We will need

**Lees' Lemma.** Let $M^n$ be a topological manifold (without boundary), $n \geq 5$; if $M^n$-point admits a PL-manifold structure, $M^n$ admits a PL-manifold structure.

**Proof of Lemma.** By the Novikov-Siebenmann theorem [5], [6], the end of $M^n$-point has a neighborhood PL-equivalent to $\Sigma^{n-1} \times R$, $\Sigma^{n-1}$ a PL-homotopy sphere. But for $n \geq 5$, $\Sigma^{n-1} \times R$ is PL-equivalent to $S^{n-1} \times R$. By taking $t$ sufficiently large, $S^{n-1} \times t$ is contained in the interior of a disc neighborhood $D^n$ of the point in $M^n$. By the Schoenflies theorem [2], $S^{n-1} \times t$ bounds a disc $D^n_t$ in $D^n$. Thus $(M^n - \text{Int } D^n_t) \cup C(S^{n-1} \times t)$ is a PL-manifold homeomorphic to $M^n$.

**Proof of A(2).** It will be sufficient to show that if either $\text{Im}_{M'}(M, Q)$ or $R(TM'/M, TQ)$ is nonempty, there is a neighborhood $V$ of $M$ in $M'$ that admits a PL-manifold structure. For then there is a compact PL-manifold $N$, with $M \subset \text{Int } N \subset V'$, $V$ any sufficiently small neighborhood of $M$. Since $N$ has a handle decomposition, we may apply Lees' theorem to $N$, and the result follows easily.

Now if $\text{Im}_{M'}(M, Q)$ is nonempty, there is an immersion $f: V \to Q$, $V$ some open neighborhood of $M$. But then $V$ admits a PL-manifold structure, since $Q$ does.

If $R(TM'/M, TQ)$ is nonempty, there is a neighborhood $U$ of $M$ and a bundle map $\psi: T'U \to TQ$. Cover $M$ by a finite number of coordinate neighborhoods $\{V_i\}$ and let $\{\overline{V}_i\}$ be a shrinking of this cover, with $\overline{V}_i \subset V_i$, $\overline{V}_i$ compact. Let $C_i = U_{i-1} \cap \overline{V}_i$. We will prove inductively,
that if \( \phi_{r-1} : U_{r-1} \to Q \) is an immersion of a neighborhood of \( C_{r-1} \), such that \( d\phi_{r-1} \) is homotopic to \( \psi \mid U_{r-1} \), then there is an immersion \( \phi_r : U_r \to Q \), where \( U_r \) is a neighborhood of \( C_r \) and \( d\phi_r \sim \psi \mid U_r \). (The result is trivial for \( C_1 = \emptyset \)).

Triangulate \( V_r \) sufficiently fine, so that any simplex of \( V_r \) that meets \( C_{r-1} \) is contained in \( U_{r-1} \). Now \( V_r \) is contained in a finite subcomplex \( K \) of \( V_r \). Now by induction over the skeletons, we can immerse a neighborhood \( W \) of \( C_{r-1} \cup K^{(k)} \), using Lemma 2 of [4], with \( n = k \), provided \( k < q \). Since \( Q \) is PL, \( W \) admits a PL-manifold structure. Thus a neighborhood \( W' \) of \( C_r \) admits a PL-structure except at a finite number of points. Therefore, \( W' \) admits a PL-structure by Lee's lemma. But then there is a compact PL-manifold \( N_r \), with \( C_r \subset \text{Int } N_r \subset W' \). By applying Lee's theorem to \( N_r \), we obtain an immersion \( \phi_r \) of a neighborhood \( U_r \) of \( N_r \) (and hence of \( C_r \)) with \( d\phi_r \), homotopic to \( \psi \).

This completes the inductive step, and hence there is an immersion \( \phi : U \to Q \), \( U \) a neighborhood of \( M \) in \( M' \). Hence \( U \) admits a PL-structure. Q.E.D.

**Proof of B.** If \( \dim M = \dim Q \), then \( M \) has a collar in \( M' \), and hence is a deformation retract of a neighborhood \( U \). It follows that any bundle map of \( TM'/M \) extends canonically to \( TM'/U \), and any two such are canonically homotopic relative to \( M \). Thus the two definitions of \( R(TM'/M, TQ) \) are equivalent.

For \( \dim M < \dim Q \), the author does not know whether a locally flat submanifold is a neighborhood deformation retract; however, it is true stably. First note that if \( E \) is an \( \mathbb{R}^n \)-bundle over a space \( X \) of the homotopy type of a locally finite simplicial complex, the total space \( E(\varepsilon) \) also has this property; and it follows that the projection \( \rho : E(\varepsilon) \to X \) and zero section \( i : X \to E(\varepsilon) \) are homotopy inverses, and \( X \) is a deformation retract of \( E(\varepsilon) \).

Now \( M \) has a normal bundle \( v \) in \( M' \times \mathbb{R}^k \), \( k \) sufficiently large, and \( M \) is a (strong) deformation retract of \( E(\rho) \). Since \( TM' \) may be lifted to a bundle \( \tau \) over \( M' \times \mathbb{R}^k \) such that \( \tau \mid M' \times 0 = TM' \), it follows easily that the two definitions of \( R(TM'/M, TQ) \) are equivalent in this case also.

**Proof of C.** Embed \( M^0 \) in \( S^{n+k} \), \( k \) sufficiently large so that \( M^0 \) has a normal \( \mathbb{R}^k \) bundle \( v \). Now \( v \mid D^* \cong D^n \times \mathbb{R}^k \). Removing the \( \text{Int}(D^n \times D^k) \) form \( S^{n+k} \), we get an embedding of \( (M^0, \partial M^0) \) in \( (D^{n+k}, \partial D^{n+k}) \) with normal bundle \( v^0 = v \mid M^0 \), since every neighborhood of the zero section of an \( \mathbb{R}^k \)-bundle contains an equivalent \( \mathbb{R}^k \)-bundle (see [3]). Note that the tangent bundle of \( E(v^0) \) is trivial, and \( E(v^0) \) is a locally finite simplicial complex dominating \( M^0 \). Pulling \( TM^0 \) back over \( E(v^0) \), we
have that \( M^n \times R^{n+k} \) admits a PL-manifold structure as a PL-manifold \( W \) with boundary.

The Novikov-Siebenmann relative splitting theorem \([1],[5],[6]\), provides a PL-manifold \( Q^n \) with \( \partial Q^n = S^{n-1} \), and a PL-homeomorphism \( h: Q^n \times R^{n+k} \to W \). Thus \( h \) defines a homotopy equivalence of pairs \( \varphi: (Q^n, \partial Q^n) \to (M^n, \partial M^n) \) such that \( TQ^n \cong \varphi^* TM^n \) as PL-bundles (actually stably isomorphic, but \( Q^n \cong (n-2) \) complex, and stably isomorphic implies isomorphic). Let \( \psi \) be a homotopy inverse of \( \varphi \); then \( \psi: (M^n, \partial M^n) \to (Q^n, \partial Q^n) \) is covered by a bundle map \( \psi_*: TM^n \to TQ^n \) of topological bundles. Let \( Q = Q^n \cup S^{n-1} \), and \( M' = M \)-point.

Then Theorem A(2) applies to produce an immersion in \( Q \) of a neighborhood \( U \) of \( M^n \) in \( M \). Thus \( U \) admits a PL-manifold structure, and by Lees' lemma, \( M \) admits a PL-structure. Q.E.D.

**Proof of D.** Let \( M \) be the underlying topological manifold of \( M_2 \), and identify it with that of \( M_1 \) via \( h \). The condition on \( dh \) implies by A(2), that \( M \times I \) may be immersed in \( M_2 \times R \) so that the immersion is PL with respect to the \( M_1 \) structure near \( M \times 0 \) and with respect to the \( M_2 \) structure near \( M \times 1 \). This gives a PL-structure on \( M \times I \) which is a concordance between the \( M_1 \) and \( M_2 \) structures. The result follows easily.

**BIBLIOGRAPHY**


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