SOME NONZERO HOMOTOPY GROUPS OF SPHERES

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1. The purpose of this note is to establish some nonzero elements in the homotopy groups of spheres. This results from unstabilizing a method of Adams. Namely, an Adams spectral sequence is used to detect elements in \( \pi_{n+i}(S^n) \) for various \( n \) and \( i \); in addition to the \( d \) and \( e \) invariants of Adams, the Hopf invariants are used to show that certain of these elements are nonzero. One consequence will be the following.

Consequence. The groups \( \pi_{4+i}(S^4) \) are nonzero for all \( i \geq 0 \).

2. Recall the mod-\( p \)-restricted lower central series spectral sequence (abbr: mod-\( p \)-RLCSSS), constructed as in [4], [5] and [10]. For each simplicial set \( X \), form \( GX \) as in [6], filter \( GX \) by its mod-\( p \)-RLCS, and pass to the homotopy exact couple. The resulting spectral sequence we will label \( E_{rsd}(X) \), where \( s = \text{filtration} \) and \( d = \text{dimension} \). The results of [4, §(2.4)] show that for the sphere spectrum \( S \), the term \( E^1(S) \) of the mod-2-RLCSS is a ring \( A \), with multiplicative generators \( \lambda_i \) for each \( i \geq 0 \). An additive basis for \( E^1(S) \) consists of all monomials \( \lambda_I = \lambda_{i_1} \cdots \lambda_{i_k} \), where \( I = (i_1, \cdots, i_k) \) is a sequence of nonnegative integers with \( 2i_j \geq i_{j+1} \) for \( j = 1, 2, \cdots, k - 1 \). Call such monomials allowable. In the unstable case, the results of [4, §(5.4)] show that for the \( n \)-sphere \( S^n \), \( E^1(S^n) \) is the subvector space of \( A \) with basis all \( \lambda_I \) which are allowable and for which \( i_1 < n \). Such a monomial \( \lambda_I \in E^1(S^n) \), where \( I = (i_1, \cdots, i_k) \), has filtration \( k \), and dimension \( n + \sum i_j \).

3. There is a short exact sequence of differential vector spaces:

\[
0 \to E^1_{s,n+i}(S^n) \to E^1_{s,n+i+1}(S^{n+1}) \to E^1_{s-1,n+i+1}(S^{2n+1}) \to 0
\]

where \( i \) is the inclusion and \( h \) is defined on the allowable basis by

\[
h(\lambda_I \lambda_{I'}) = \lambda_I \quad \text{for} \quad j = n,
\]

\[
 = 0 \quad \text{for} \quad j < n.
\]

From this, there derives a long exact sequence

\[
\cdots \to E^2(S^n) \to E^2(S^{n+1}) \xrightarrow{h_*} E^2(S^{2n+1}) \to \cdots.
\]

It can be shown that \( h_* \) commutes with all differentials, and is induced
by the Hopf-invariant in the SHP-sequence of Whitehead, James:

\[
\cdots \to \pi_{n+1}(X) \xrightarrow{S} \pi_{n+i+1}(S^{n+1}) \xrightarrow{H} \pi_{n+i+1}(S^{2n+1}) \xrightarrow{P} \cdots.
\]

From the sequence (3.1), some calculations in \(E^2(S^n)\) can easily be made.

4. For each \(m \geq 0\), define functions \(\phi_2(m), \phi_3(m), \phi_4(m), \phi_5(m), \phi(m)\) by the rules:

\[
\begin{array}{c|cccccccc}
 m = 8k+ & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
\phi_2(m) = 4k+ & 0 & 1 & 2 & 3 & 4 & 4 & 5 & 4 \\
\phi_3(m) = 4k+ & 0 & 1 & 2 & 3 & 3 & 4 & 3 & 4 \\
\phi_4(m) = 4k+ & 0 & 1 & 2 & 3 & 4 & 4 & 4 & 4 \\
\phi_5(m) = 4k+ & 0 & 1 & 2 & 3 & 3 & 4 & 4 & 4 \\
\phi(m) = 4k+ & 0 & 1 & 2 & 3 & 3 & 3 & 3 & 4 \\
\end{array}
\]

The function \(\phi(m)\) describes the Adams vanishing line: \(\text{Ext}^{s,t}_A(Z_2, Z_2) = 0\) for \(s > \phi(t-s)\). Unstably, the functions \(\phi_n(m)\) (set \(\phi_n(m) = \phi(m)\) for \(n \geq 6\)) also describe a vanishing line, possibly modulo a tower, as follows.

**Theorem.** \(E^2_{i,n+i}(S^n) = 0\) for \(s > \phi_n(i)\), except for the tower at \(i = 0\), and the tower which occurs when \(n\) is even and \(i = n-1\).

This can be proven using the stable vanishing line \(\phi(m)\) of Adams [1], (3.1), and downward induction.

**Corollary.** In the 2-component of \(\pi_{n+i}(S^n)\), each element has order \(\leq 2^{\phi_n(i)}\).

This is of course the unstable analogue of [1, p. 69]. There is also a similar vanishing line for each prime \(p\), and all together give a bound for the order of any element (of finite order).

5. Let \(P\) be the periodicity operator defined by the Massey product \(P(x) = \{x, \lambda_0^*, \lambda_7^*\}\). The following table describes some (not all) non-zero elements in \(E^2(S^n)\) near the vanishing line. They are cycles in every \(E^r(S^n)\) for which they are defined, as the differentials on them land in the vanishing-zone or in a tower.
The elements $P^{k-1}(c_0)$, $P^{k-1}(h_2c_0)$, $P^{k}(h_7)$, $P^{k}(h_2)$, $P^{k}(h_0h_2)$, $P^{k}(h_0^2h_2)$, $P^{k}(h_0^2h_3)$ are shown never to be boundaries in the stable Adams spectral sequence because of nonzero $d$ or $e$ invariants; see [2], [7], [8], [9]. Hence, by naturality of suspension, their precursors are never boundaries in each $E^r(S^n)$ of the mod-2-RLCSSS.

The Hopf-invariant $h_*: E^r(S^n) \rightarrow E^r(S^n)$ shows that the elements $P^{k}(\lambda_3\lambda_1^2)$, $P^{k}(\lambda_3^3\lambda_1)$ are not boundaries in any $E^r(S^n)$, since $h_*$ of them are not boundaries in $E^r(S^n)$. Similarly, the elements $P^{k}(\lambda_3\lambda_1)$, $P^{k}(\lambda_3^3\lambda_1)$ and $P^{k}(\lambda_3^3\lambda_1^2)$ are never boundaries in any $E^r(S^n)$.

6. For odd primes $p$, the $E^1$-term of the mod-$p$-RLCSSS for odd spheres is described in [4, §8]. The analogous vanishing statement is $E^2_{i,p+1}(S^n) = 0$, for all odd $n$, and $s > [i+3/2p-2]$. Also, in filtration $k$ and dimension $3+2k(p-1)-1$, $E^3(S^n)$ has a single generator say $a_k$. As all differentials on $a_k$ land in the vanishing zone, $a_k$ is a permanent cycle; also, $a_k$ is never a boundary, shown by a mod-$p$ version of [9]. Thus $a_k$ detects a nonzero class of order $p$ in $\pi_{3+2k(p-1)-1}(S^n)$. Of course the element detected by $a_k$ is just (a nonzero multiple of) Toda's $\alpha_k$ shown to be nonzero by Adams' $e$-invariant argument.

7. It is now easy to exhibit some nonzero homotopy classes, as each of the elements in the table detects a nonzero class in $\pi_{w}(S^n)$ for the corresponding value of $n$. Using also the elements $\alpha_k(3)$ for stems

<table>
<thead>
<tr>
<th>Stem dim $i$</th>
<th>Filtration $s$</th>
<th>Minimum value of $n$</th>
<th>Element in $E^k(S^n)$</th>
<th>Stable element in $Ext(\mathbb{Z}_2, \mathbb{Z}_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$8k$</td>
<td>$4k - 1$</td>
<td>3</td>
<td>$P^{k-1}(\lambda_3\lambda_1^2)$</td>
<td>$P^{k-1}(c_0)$</td>
</tr>
<tr>
<td>$8k + 1$</td>
<td>$4k$</td>
<td>2</td>
<td>$P^{k-1}(\lambda_3\lambda_1^3)$</td>
<td>$P^{k-1}(h_2c_0)$</td>
</tr>
<tr>
<td>$8k + 1$</td>
<td>$4k + 1$</td>
<td>3</td>
<td>$P^{k}(\lambda_1)$</td>
<td>$P^{k}(h_7)$</td>
</tr>
<tr>
<td>$8k + 2$</td>
<td>$4k + 2$</td>
<td>2</td>
<td>$P^{k}(\lambda_1^2)$</td>
<td>$P^{k}(h_1)$</td>
</tr>
<tr>
<td>$8k + 3$</td>
<td>$4k + 1$</td>
<td>5</td>
<td>$P^{k}(\lambda_1)$</td>
<td>$P^{k}(h_2)$</td>
</tr>
<tr>
<td>$8k + 3$</td>
<td>$4k + 2$</td>
<td>3</td>
<td>$P^{k}(\lambda_1^3)$</td>
<td>$P^{k}(h_0h_2)$</td>
</tr>
<tr>
<td>$8k + 3$</td>
<td>$4k + 3$</td>
<td>2</td>
<td>$P^{k}(\lambda_1^4)$</td>
<td>$P^{k}(h_0^2h_2)$</td>
</tr>
<tr>
<td>$8k + 4$</td>
<td>$4k + 3$</td>
<td>4</td>
<td>$P^{k}(\lambda_1^5)$</td>
<td>0</td>
</tr>
<tr>
<td>$8k + 5$</td>
<td>$4k + 3$</td>
<td>3</td>
<td>$P^{k}(\lambda_1^6)$</td>
<td>0</td>
</tr>
<tr>
<td>$8k + 6$</td>
<td>$4k + 4$</td>
<td>4</td>
<td>$P^{k}(\lambda_1^7)$</td>
<td>0</td>
</tr>
<tr>
<td>$8k + 7$</td>
<td>$4k + 4$</td>
<td>5</td>
<td>$P^{k}(\lambda_1^8)$</td>
<td>$P^{k}(h_0^2h_3)$</td>
</tr>
</tbody>
</table>
\[ \equiv 7 \pmod{8}, \text{ there follows consequence (1). Further, } \pi_{2+i}(S^3) \text{ is nonzero at least for all } i \not\equiv 6 \pmod{8}, \text{ and hence also } \pi_{2+i}(S^3) \text{ is nonzero at least for all } i \not\equiv 7 \pmod{8}. \]

**BIBLIOGRAPHY**