The aim of this note is to describe the structure of a class of non-commutative rings which possess a variant of the Euclidean algorithm and indicate some properties of such rings.

All rings are associative and possess unity; subrings and homomorphisms are unitary. A domain is a (not necessarily commutative) ring without nonzero zero-divisors.

Let $\mathcal{R}$ be a ring and $\phi$ be an ordinal-valued function defined on $\mathcal{R}\sim(0)$. Put $\phi(0) = -\infty$ and let $(-\infty) + (-\infty) = \alpha + (-\infty) = (-\infty) + \alpha = -\infty$ and $-\infty < \alpha$ for every ordinal $\alpha$ in the range of $\phi$. $\phi$ is called a transfinite left division algorithm on $\mathcal{R}$ if, for all $a, b \in \mathcal{R}$, the following conditions hold:

1. $\phi(a - b) \leq \max\{\phi(a), \phi(b)\}$,
2. $\phi(ab) = \phi(b) + \phi(a)$,
3. if $b \neq 0$, then there exist $q, r \in \mathcal{R}$ such that $a = qb + r$, $\phi(r) < \phi(b)$.

Clearly, every ring with a transfinite left division algorithm is a left principal ideal domain.

We need some terminology and notations. Let $\rho$ be a mono-endomorphism of a domain $D$. A mapping $\delta : D \rightarrow D$ is called a $\rho$-derivation on $D$ if $\delta(a + b) = \delta(a) + \delta(b)$ and $\delta(ab) = \rho(a)\delta(b) + \delta(a)b$ hold for all $a, b \in D$.

Let $D$ be a subdomain of a domain $R$. Let $x$ be an element of $R$ such that every nonzero element $r \in R$ can be uniquely expressed as $\sum_{i=0}^{n} d_i x^{n_i}$ where $d_i \in D \sim(0)$ and $n_i$ are integers with $0 \leq n_0 < \cdots < n_s$. Further, suppose that there exists a mono-endomorphism $\rho$ of $D$ and a $\rho$-derivation $\delta$ on $D$ such that $xd = \rho(d)x + \delta(d)$ for all $d \in D$.

This situation is expressed symbolically as $R = D[x, \rho, \delta]$. Let $R$ be a domain, $\alpha$ a nonzero ordinal and $\{R_\beta : \beta < \alpha\}$ a set of subdomains of $R$ such that

1. $R = \bigcup_{\beta < \alpha} R_\beta$,
2. if $0 < \beta < \alpha$ then $R_\beta = (\bigcup_{\gamma < \beta} R_\gamma)[x_\beta, \rho_\beta, \delta_\beta]$. We express this situation symbolically as $R = R_0[x_\beta, \rho_\beta, \delta_\beta : 0 < \beta < \alpha]$. Thus, $\bigcup_{\gamma < \beta} R_\gamma = R_0[x_\gamma, \rho_\gamma, \delta_\gamma : 0 < \gamma < \beta]$. If all $\delta_\beta$ are zero derivations, we simplify the notation and put $R = R_0[x_\beta, \rho_\beta : 0 < \beta < \alpha]$.

**Theorem 1** (Cf. [2], [4]). A ring $R$ has a transfinite left division algorithm if and only if $R = K[x_\beta, \rho_\beta, \delta_\beta : 0 < \beta < \alpha]$, where $K$ is a skew
field and, for every $0 < \beta < \alpha$,
\[
\rho_\beta(K[x_\gamma, \rho_\gamma; 0 < \gamma < \beta]) \subseteq K.
\]

A construction is given to prove the following

**Theorem 2.** Let $k$ be an arbitrary skew field and $\alpha$ be an arbitrary nonzero ordinal. There exists a skew field $K$ containing $k$ as a subskew field and a ring $R = K[x_\beta, \rho_\beta; 0 < \beta < \alpha]$ such that
\[
\rho_\beta(K[x_\gamma, \rho_\gamma; 0 < \gamma < \beta]) \subseteq K.
\]

For $\alpha = 1$, any skew field would do. For $\alpha = 2$, $K[x, id]$ works. For $\alpha = 3$, Theorem 2 already contains a counterexample to a conjecture of I. N. Herstein, stated as highly likely to be true [3, p. 75]. For other implications, see [5].

In the following two theorems, $K$ is a skew field and
\[
R = K[x_\beta, \rho_\beta; 0 < \beta < \alpha]
\]

where, for $0 < \beta < \alpha$,
\[
\rho_\beta(K[x_\gamma, \rho_\gamma; 0 < \gamma < \beta]) \subseteq K.
\]

**Theorem 3** (Cf. [1]). $R$ is a right primitive ring. $R$ is a left primitive ring if and only if $\alpha$ is a nonlimit ordinal.

**Theorem 4.** Let $\Omega_\lambda$ be the first ordinal of cardinality $\aleph_\lambda$. We have
\[
\text{r. gl. dim } R = \infty \quad \text{if } \alpha \geq \Omega_\omega,
\]
\[
\geq n + 2 \quad \text{if } \alpha \geq \Omega_\omega + 1 \text{ where } n < \omega,
\]
\[
\geq 2 \quad \text{if } \omega \geq \alpha > 2.
\]

Using Theorems 2 and 4 it is shown that there exist rings with a transfinite left division algorithm having a prescribed right global dimension. Notice that the left homological dimension of any such ring is either 0 or 1 (cf. [6 and references given there]).

In a slightly different direction, we have

**Theorem 5.** A domain $R = D[x_\beta, \rho_\beta; 0 < \beta < \alpha]$ is a left principal ideal domain if and only if $D$ is a left principal ideal domain and, for every $0 < \beta < \alpha$,
\[
\rho_\beta(D[x_\gamma, \rho_\gamma; 0 < \gamma < \beta] \sim (0)) \subseteq U(D)
\]

where $U(D)$ is the group of units of $D$.

**Acknowledgement.** The author is thankful to Professor N. Greenleaf for encouragement and advice.
REFERENCES


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