ON DISCRETE BOREL SPACES AND
PROJECTIVE SETS

BY B. V. RAO

Communicated by David Blackwell, December 3, 1968

Let $I$ denote the unit interval, $S = I \times I$ the unit square; $C_I$ and
$C_S$ the class of all subsets of $I$ and $S$, respectively. By $C_I \times C_I$ is
meant the $\sigma$-algebra on $S$ generated by rectangles with sides in $C_I$.
The purpose of this note is to prove the following theorem (which
settles a problem of S. M. Ulam) and observe some of its conse­
quences. Without explicit mention, the axiom of choice has been assumed
throughout this paper. CH stands for the continuum hypothesis.

**Theorem 1.** If CH is valid, then $C_I \times C_I = C_S$.

**Proof.** First, observe that if $f$ is any function defined on a subset
of $I$ into $I$ then its graph

$$G = \{(x, y): x \in \text{Domain of } f, f(x) = y\}$$

is in $C_I \times C_I$. For this it suffices to verify that

$$G = \bigcap_{n=1}^{\infty} S_n; \quad S_n = \bigcup_{k=1}^{n} \{A_{nk} \times B_{nk}\},$$

$$A_{nk} = \{x \in \text{Domain } f: (k - 1)/n \leq f(x) < k/n\},$$

$$B_{nk} = \{y \in \text{Range } f: (k - 1)/n \leq y < k/n\}.$$  

(For $k = n$; include the right endpoint as well.)

Second, if $B \subset S$ be such that every vertical section is at most
countable then $B \in C_I \times C_I$. This follows by realizing $B$ as countable
union of graphs.

Third, if $B \subset S$ is such that every horizontal section is at most
countable then $B \in C_I \times C_I$.

Fourth, $S = X \cup Y$ where every vertical section of $X$ is at most
countable and every horizontal section of $Y$ is at most countable [4].
This can be done by realizing $I$ as the set of ordinals less than the
first uncountable ordinal (by using CH) and then taking the portions
below and not below the diagonal.

Finally, if $B \subset S$ then by previous remarks $B \cap X, B \cap Y$ are in
$C_I \times C_I$ to complete the proof.

Let $Z$ be a set of cardinality $\aleph_1$, the first uncountable cardinal. An
obvious modification of the above theorem gives us

**Theorem 2.** The product of discrete \( \sigma \)-algebras on \( Z \) is the discrete \( \sigma \)-algebra on \( Z \times Z \). Consequently if \( A \subseteq S \) be such that \( \text{Card} (A) \leq N_1 \), then \( A \subseteq C_T \times C_T \).

Clearly, Theorem 1 is a consequence of the above theorem together with CH.

**Theorem 3.** Let \( \{ Z_\alpha, \alpha \in T \} \) be any collection of subsets (possibly empty also) of \( Z \) where \( \text{Card} (T) = N_1 \). Then there is a separable (countably generated and containing all singletons) \( \sigma \)-algebra on \( Z \) containing the given collection.

**Proof.** There is no loss in taking \( T = Z \), as we do. Put

\[
A = \bigcup_{\alpha \in Z} \{ \alpha \times Z_\alpha \}.
\]

By Theorem 2, \( A \) is in the product of discrete \( \sigma \)-algebras on \( Z \) and consequently it is in the \( \sigma \)-algebra generated by a countable number of rectangles, say, \( \{ A_i \times B_i, i \geq 1 \} \). Any separable \( \sigma \)-algebra on \( Z \) (clearly there are such) containing \( \{ A_i, B_i, i \geq 1 \} \) will suffice for our purpose.

As an immediate consequence of the above theorem we have the following which gives an affirmative answer to a question of S. M. Ulam [6], and disproves a conjecture of the author [2].

**Theorem 4.** Let CH be valid. Then there is a separable \( \sigma \)-algebra on \( I \) containing all the analytic sets of \( I \). In fact there is one such containing all projective [3] sets of \( I \).

In the terminology of Szpilrajn-Marczewski [1] the above theorem can be restated as

**Theorem 5.** Let CH be valid. Then there is a one to one transformation \( \phi \) of \( I \) into \( I \), transforming each set projective in \( I \) into a set Borel in \( \phi (I) \).

We now formulate a generalization of the notion of projective sets and solve a related problem of Ulam [7]. Let \( C = \{ A_p; p \in T \} \) be a collection of subsets of \( I \) where \( \text{Card} (T) = c \). The projections on \( I \) of sets of the \( \sigma \)-algebra on \( S \) over the rectangles \( A_p \times A_q \) with sides in \( C \), constitute \( P_1 \), the first projective class. Having defined \( P_\alpha \) for \( \alpha < \gamma < \Omega \) we define \( P_\gamma \), as the projections on \( I \) of the sets of the \( \sigma \)-algebra on \( S \), over the rectangles with sides in the previous projective classes. These are called generalized projective sets. Clearly one
need not proceed after the first uncountable ordinal $\Omega$. Since each $P_\alpha$ has cardinality not greater than $c$, we have

**Theorem 6.** Let $CH$ be valid. Then there is a separable $\sigma$-algebra on $I$ containing all its generalized projective sets over any fixed class $C$, where of course $\text{Card } (C) \leq c$.

The author [2] has proved the following theorem, elsewhere.

**Theorem 7.** Let $L$ be any class of subsets of $I$ which are measurable w.r.t. a fixed nonatomic probability measure on the Borel subsets of $I$. Then $L$ does not contain any separable $\sigma$-algebra including all analytic subsets of $I$.

In view of Theorems 7 and 4, one has

**Theorem 8.** Let $CH$ be valid. Fix any separable $\sigma$-algebra on $I$, say $A_0$, containing all the analytic subsets of $I$. For every nonatomic probability measure on the Borel field of $I$, there is at least one nonmeasurable set in $A_0$.

**Theorem 9.** There exists a separable $\sigma$-algebra on $Z$ which supports no continuous probability measure.

**Proof.** Observe, following Ulam [5], that with each finite ordinal $n$ and countable ordinal $\alpha$ we can associate a subset $K(n, \alpha)$ of $Z$ satisfying the following:

(i) for each fixed $\alpha$, $\bigcup_n K(n, \alpha)$ is a cocountable subset of $Z$, and
(ii) for each fixed $n$, $\{K(n, \alpha) : \alpha \text{ countable ordinal}\}$ is a disjoint family.

Take any separable $\sigma$-algebra on $Z$ containing all these sets (assured by Theorem 3). The argument of Ulam [5] now completes the proof.

If $CH$ is assumed, the above theorem says that on $I$ there is a separable $\sigma$-algebra which does not support a continuous probability measure. If one wishes, this $\sigma$-algebra can be taken to contain all Borel sets or all analytic subsets of $I$.

The author expresses his appreciation and thanks to Dr. Ashok Maitra for the inspiring discussions and useful comments. Thanks are also due to him for many suggestions on an earlier version of this paper.

**References**

Let $E$ be an $(n-1)$-sphere bundle over a base space $B$, with the orthogonal group as structural group. By an almost-complex structure on $E$ we mean a reduction of the structural group to the unitary group. By an $A$-structure on $E$ I mean a fibre-preserving map $f: E\to E$ such that $fx$ is orthogonal to $x$ for all $x\in E$. For example, an almost-complex structure determines such a map through the action of the scalar $J$ such that $J^2 = -1$. Note that $n$ must be even if an $A$-structure exists. When $E$ is trivial this necessary condition is also sufficient.

I describe $E$ as homotopy-symmetric if $1\equiv u: E\to E$, by a fibre-preserving homotopy, where $u$ denotes the antipodal map given by $ux = -x$. This condition also implies that $n$ is even. An $A$-structure $f$ on $E$ determines a fibre-preserving homotopy $f_t$ ($t\in I = [0, 1]$), where $f_t x = x \cos \pi t + f(x) \sin \pi t$, and so $E$ is homotopy-symmetric. I assert that the converse holds in the stable range, so that we have

**Theorem 1.** Let $B$ be a finite complex such that $\dim B \leq n-4$. Then $E$ admits an $A$-structure if and only if $E$ is homotopy-symmetric.

A proof can be given as follows. Let $p: E\to B$ denote the fibration. Let $E'$ denote the space of pairs $(x, y)$, where $x, y\in E$, such that $px = py$ and such that $x$ is orthogonal to $y$. We fibre $E'$ over $E$ with projection $p'$ given by $p'(x, y) = x$. An $A$-structure $f$ on $E$ determines a cross-section $f': E\to E'$, where $f' x = (x, fx)$, and conversely a cross-section determines an $A$-structure. Let $E''$ denote the space of paths $\lambda$ in $E$ such that $p\lambda$ is stationary in $B$ and such that $\lambda(0) = \lambda(1)$. We

---

1. Research partly supported by the National Science Foundation.
2. We recall that the centre of the structural group acts on the bundle.
3. The stable range, in relation to this problem, is not quite as extensive as the stable range of ordinary theory.