CATEGORIES OF V-SETS

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Let $V$ be a partially ordered set. Then a $V$-set is a function $A: X \to V$ from a set $X$ to $V$. $V$ is the set of values for $A$, and $X$ is the carrier of $A$. If $B: Y \to V$ is another $V$-set, a morphism $f: A \to B$ is a function $\bar{f}: X \to Y$ such that $A(x) \leq B(\bar{f}(x))$ for each $x \in X$. The category of all $V$-sets is denoted $\mathcal{S}(V)$. The carrier functor $K: \mathcal{S}(V) \to \mathcal{S}$ assigns $X$ to $A: X \to V$ and $\bar{f}: X \to Y$ to $f: A \to B$, where $\mathcal{S}$ is the category of sets. See [2].

If $V$ has one point, $\mathcal{S}(V) = \mathcal{S}$. If $V = \{0, 1\}$, where $0 < 1$, $\mathcal{S}(V)$ is the category of pairs $(X, A)$ of sets, where $A \subseteq X$. If $V$ is the closed unit interval, $\mathcal{S}(V)$ is the category of “fuzzy sets”, as used by Zadeh and others [1], [5] for problems of pattern recognition and systems theory. When $V$ is a Boolean algebra, $V$-sets are Boolean-valued sets, as used by Scott and Solovay for independence results in set theory (however, their notion of morphism is different).

If $V$ is complete, $\mathcal{S}(V)$ is a pleasant category satisfying all Lawvere’s axioms [3] for $\mathcal{S}$ except choice, modulo some substitutions of the $V$-set with carrier 1 and value 0 for the terminal object. In particular,

**Theorem 1.** If $V$ is complete, $\mathcal{S}(V)$ is complete and cocomplete, has an exponential (i.e., a coadjoint to product) and a “Dedekind-Pierce object” (i.e., an object which looks like the set of integers; see [3]).

Let $\text{Poc}$ denote the category of partially ordered classes, and let $\mathcal{L}$ be a subcategory of $\text{Poc}$. Then a category $\mathcal{C}$ is $\mathcal{L}$-ordered if the power function $\mathcal{P}: |\mathcal{C}| \to \text{Poc}$ factors through $\mathcal{L}$, where $\mathcal{P}(A)$ is the class of all equivalence classes of monics with codomain $A (f \equiv g$ if $\exists$ an isomorphism $h$ such that $fh = g$). Denote the image of $A \sqcup B$ by $f(A)$, and the image of the composite $A \sqcup A \sqcup B$, where $i$ is monic, by $f(A')$. Then $\mathcal{C}$ has associative images if it has images such that $f(g(A)) = (fg)(A)$, whenever $A \sqcup B \sqcup C$. $\mathcal{P}$ can be construed as a functor when $\mathcal{C}$ has associative images. Let $\text{CL}$ denote the category of complete lattices, and call a category $\mathcal{C}_i$ if a coproduct of monics is always monic.

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THEOREM 2. A CL-ordered category with associative images has equalizers, inverse images, unions, intersections, and epic images. If it has coproducts, it is $C_1$.

An object $P$ in a category $\mathcal{C}$ is monic if every arrow $P \to A$ is monic, and is further atomic if every $P \to A$ is atomic in $\varphi(A)$. $P$ is good if the functor $[P, -] : \mathcal{C} \to \mathbb{S}$ is noninitial preserving. A union $\bigcup_i A_i$ in $\mathcal{C}$ is disjoint if $i \neq j \Rightarrow A_i \cap A_j = \emptyset$, where $\emptyset$ is the initial object. Let $CDL$ be the category of completely distributive lattices, i.e., complete lattices satisfying the law $a \land V b_i = V_i(a \land b_i)$. Such lattices $V$ have pseudo-complement operators $*: V \to V$ defined by $a* = V\{b \mid a \land b = 0\}$. Call $V \in |CDL|$ disjointed if for each pair $x, y$ of unequal atoms, $x* \lor y* = 1$, the maximal element of $V$, and call $\mathcal{C}$ disjointedly $CDL$-ordered if each $\varphi(A) \in |CDL|$ is disjointed.

THEOREM 3. A category $\mathcal{C}$ is equivalent to $\mathbb{S}(V)$ for some $V \in |CDL|$ if and only if:

1. $\mathcal{C}$ has an atomic monic good projective generator $P$;
2. $\mathcal{C}$ has initial and terminal objects, $\emptyset$ and $I$, respectively;
3. $\mathcal{C}$ has coproducts, which are disjoint unions; and conversely, each disjoint union in $\mathcal{C}$ is a coproduct in $\mathcal{C}$;
4. $\mathcal{C}$ has associative images;
5. $\mathcal{C}$ is disjointedly $CDL$-ordered; and
6. $P \sqcup P$ is not isomorphic to $P$.

The Axioms (1)–(6) are easily verified for $\mathbb{S}(V)$, $V \in |CDL|$. We now sketch the converse, which (surprisingly) makes no use of adjoint functors. Essential use is made of Theorem 2, via Axioms (4) and (5).

Call the elements of $[P, A]$ the points of $A$. We first show the one-pointed objects of $\mathcal{C}$ are the subobjects of $I$, except $\emptyset$; denote this lattice $V$. A calculation shows that each $A \subseteq \mathcal{C}$ is a disjoint union $\bigcup_{x \in [P, A]} x^*$, so by Axiom (3), $A = \bigcup_{x \in [P, A]} x^*$. These facts combine to show that each $A$ is a subobject of $I_{[P, A]}$, the coproduct of $I$ over the index set $[P, A]$. We then show the arrows $f : A \to B$ in $\mathcal{C}$ are in 1-1 correspondence with appropriate arrows $j : [P, A] \to [P, B]$ in $\mathbb{S}$. The functor $E : \mathcal{C} \to \mathbb{S}(V)$ defined by $K(E(A)) = [P, A], E(A)(x) = x^* \subseteq V$, and $E(f) = [P, f]$, is then shown to be full, faithful, and representative.

The addition to Axioms (1)–(6) of either the categorical axiom of choice, or the condition $I = P$, yields a characterization of $\mathbb{S}$. For finite distributive lattices $V$, categories of $V$-sets with finite carrier are similarly characterized by all elementary axioms.
REFERENCES


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ERRATUM

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Page 125:
Line 3. $\tilde{\phi}^p / f^p$ should read $\tilde{\phi}^p / f^p$.
Line 9. $\alpha(t^p_m) = s^t_m$ should read $\alpha(t^p_m) = s^t_m - \sum_{q=1}^{m-1} a_q t^p q$.
Line 10 from bottom. $C^N$ should read $C^n$. 