SOME LINEAR TOPOLOGICAL PROPERTIES OF L^∞ OF A FINITE MEASURE SPACE

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We are interested here in isomorphic invariants of the various Banach spaces associated with the spaces $L^\infty(\mu)$ for finite measures μ. (Throughout, "μ" and "ν" denote arbitrary finite measures on possibly different unspecified measureable spaces.) We classify the spaces $L^\infty(\mu)$ themselves up to isomorphism (linear homeomorphism) in §3, where we also obtain information on the spaces A and A^* for subspaces A of $L^1(\mu)$. In §2, we give a short proof of a result (Corollary 2.2) which simultaneously generalizes the result of Pełczyński that $L^1(\mu)$ is not isomorphic to a conjugate space if μ is nonpurely atomic [7], and the result of Gel'fand that $L^1[0, 1]$ is not isomorphic to a subspace of a separable conjugate space (c.f. [8]). We also obtain there that an injective double conjugate space is either isomorphic to l^∞ or contains an isomorph of $l^\infty(\Gamma)$ for some uncountable set Γ, if it is infinite dimensional. (Henceforth, all Banach spaces considered are taken to be infinite dimensional. Also, we recall that a Banach space is called injective if every isomorphic imbedding of it in an arbitrary Banach space Y is complemented in Y.)

We include brief proofs of some of the results. Full details of the work announced here and other related work will appear in [11].

1. **Preliminary results.** $M(S)$ denotes the space of all regular bounded scalar-valued Borel measures on S. (Throughout, "S" denotes an arbitrary compact Hausdorff space.)

LEMMA. *Let A be a closed subspace of $M(S)$. Then either there exists a positive $\mu \in M(S)$ such that $A \subset L^1(\mu)$ (that is, every member of A is absolutely continuous with respect to μ), or A contains a subspace complemented in $M(S)$ and isomorphic to $l^1(\Gamma)$ for some uncountable set Γ.*

It is easily seen that these possibilities are mutually exclusive. (In fact it follows from known results that for uncountable Γ, $l^1(\Gamma)$ is not isomorphic to a subspace of any WCG Banach space as defined in §2.)

The lemma is proved by using the Radon-Nikodým theorem and a generalization of an argument of Köthe [5]. A consequence of its proof is the

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COROLLARY. *Let Γ be an infinite set, X a Banach space, and suppose that $c_0(\Gamma)$ is isomorphic to a subspace of X^*. Then $l^1(\Gamma)$ is isomorphic to a complemented subspace of X (and consequently $l^\infty(\Gamma)$ is isomorphic to a subspace of X^*).*

If Γ is countable, this result is known and due to Bessaga and Pełczyński (Theorem 4 of [2]).

2. **Conjugate Banach spaces isomorphic to complemented subspaces of $L^1(\lambda)$.** The Banach space X is said to *satisfy the Dunford Pettis property* (X satisfies DP) if every weakly compact operator from X to an arbitrary Banach space Y maps weak Cauchy sequences into norm-Cauchy sequences. X is said to be *weakly compactly generated* (WCG) if there is a weakly compact subset of X with linear span norm-dense in X.

THEOREM 2.1. *Let the Banach space X satisfy DP. Then if X is isomorphic to a subspace of a weakly compactly generated conjugate Banach space, every weak Cauchy sequence in X converges in the norm topology of X.*

This generalizes a result of Grothendieck (cf. Proposition 1.2 of [10]).

PROOF. Let (x_n) be a sequence in X with $x_n \to 0$ weakly. It suffices to show that $x_n \to 0$ in norm. If this does not happen, then by passing to a subsequence if necessary, we may assume there is a $\delta > 0$ with $\|x_n\| > \delta$ for all n. Now we may assume that there is a Banach space B with $X \subset B^*$, with B^* WCG. Choose $b_n \in B$ with $\|b_n\| = 1$ and $|x_n(b_n)| > \delta$ for all n. Then since B^* is WCG, the unit cell of B^{**} is weak* sequentially compact (cf. Corollary 2 of [1]); thus there is a subsequence (b_{n_i}) of the b_n's and a b^{**} in B^{**} with $b^*(b_{n_i}) \to b^{**}(b^*)$ for all $b^* \in B^*$. Thus (b_{n_i}) is a weak Cauchy sequence; defining $f_j(x) = x(b_{n_j})$ for all j and $x \in X$, (f_j) is a weak Cauchy sequence in X^*. Then it follows from a result of Grothendieck (p. 138 of [4]) that $f_j(x_{n_j}) \to 0$, a contradiction. Q.E.D.

We note that $L^1(\mu)$ is WCG since $L^2(\mu)$ injects densely into $L^1(\mu)$, and it is known that $L^1(\lambda)$ satisfies DP for any measure λ. Moreover, a complemented subspace of a WCG Banach space (a space satisfying *DP*) is also WCG (satisfies DP). We thus obtain

COROLLARY 2.2 *Let λ be an arbitrary (possibly infinite) measure, and let X be a complemented subspace of $L^1(\lambda)$. Then if X is isomorphic to a subspace of a WCG conjugate Banach space, weak Cauchy sequences in X are norm-convergent and X is separable (and consequently isometric to a complemented subspace of $L^1[0, 1]$).*

The proof follows immediately from the above observations, Theorem 2.1, and a suitable version of the lemma of §1 for arbitrary subspaces A of $L^1(\lambda)$.

Corollary 2.2 has as a consequence

THEOREM 2.3. *Let B be an injective Banach space which is isomorphic to a double conjugate Banach space. Then either B is isomorphic to l^∞ or there exists an uncountable set Γ with $l^\infty(\Gamma)$ isomorphic to a subspace of B.*

Theorems 2.1 and 2.3 show that if X^* is injective and X is isomorphic to a subspace of a WCG Banach space, then if X is nonseparable or if X contains a sequence converging to zero weakly but not in norm, X is not isomorphic to a conjugate Banach space.

3. Classification of the linear isomorphism types of the space $L^\infty(\mu)$. The following result is crucial to our classification theorem, and generalizes the following (unpublished) result due jointly to W. Arveson and the author: if $L^1(\mu)$ is nonseparable, then $(L^\infty(\mu))^*$ is not separable in its weak* topology.

For a normed linear space Y, dim Y denotes the smallest cardinal number $\mathfrak{m}$ for which there exists a subset of cardinality $\mathfrak{m}$ with linear span dense in Y.

THEOREM 3.1. *Let A be a closed subspace of $L^1(\nu)$ for some ν, and let* $\dim A=\mathfrak{m}$. *Let B be a closed subspace of A^{**} such that B is isomorphic to a subspace of some WCG Banach space, and suppose that B is weak* dense in A^{**}. Then* $\dim B\geqq\mathfrak{m}$.

PROOF. By a result of Dixmier [3], there exists an S and a $\mu\in M(S)$ satisfying the following properties:

(1) for all nonempty open $U\subset S$, $\mu(U)=\mu(\overline{U})>0$, and $\overline{U}$ is open;

(2) $C(S)=L^\infty(\mu)$;

(3) $L^1(\mu)$ is isometric to $L^1(\nu)$.

(S is nothing more than the Stone space of the measure algebra of ν; in the terminology of [3], S is hyperstonian and μ is normal on S.) We may assume that $A\subset L^1(\mu)$ and that B is a subspace of $A^{\perp\perp}$ such that $B^\perp\cap C(S)=A^\perp$. (A^{**} is here identified with $A^{\perp\perp}\subset C(S)^*$ and $L^1(\mu)$ is regarded as being a subspace of $M(S)=C(S)^*$.) We assume that $\dim B<\mathfrak{m}$ and argue to a contradiction.

By the lemma of §1, there is a positive $\nu_1\in M(S)$ such that $B\subset L^1(\nu_1)$. By the Lebesgue decomposition theorem, there exist positive λ and ρ in $M(S)$ with $\nu_1=\lambda+\rho$ with ρ absolutely continuous with respect to μ, and a Borel measureable set E such that $\mu(E)=\lambda(\sim E)=0$. But then $\mu(\overline{E})=0$ also, by (1). Moreover, ν_1 is absolutely continuous with respect to $\lambda+\mu$, so we may assume that $B\subset L^1(\lambda+\mu)$.

Now we may choose a clopen (closed and open) set $U\subset\sim\overline{E}$ such that $\dim A_1>\dim B_1$, where $A_1=\{\chi_U a: a\in A\}$ and $B_1=\{\chi_U b: b\in B\}$. Thus since $B_1\subset L^1(\mu|U)$ and $\overline{B}_1\neq\overline{A}_1$, we have by (1), (2), and the Hahn-Banach theorem that there is an $f\in C(S)$ supported on U and an $a\in A$ with $\int afd\mu\neq 0$, while $\int bfd\mu=0$ for all $b\in B$. Thus $f\notin A^\perp$ yet $f\in B^\perp$, a contradiction. Q.E.D.

Our next theorem is the main result of this paragraph.

THEOREM 3.2. *$L^\infty(\mu)$ is isomorphic to $L^\infty(\nu)$ if and only if* $\dim L^1(\mu)=\dim L^1(\nu)$.

The "only if" part follows easily from Theorem 3.1; to prove the "if" part, we show using Maharam's theorem [6] that if $\dim L^1(\mu)=\dim L^1(\nu)$, then $L^1(\nu)$ is isometric to a quotient space of $L^1(\mu)$; thus $L^\infty(\mu)$ and $L^\infty(\nu)$ are of the same linear dimension, and hence isomorphic by a result of Pełczyński [9].

REMARK. Letting $\mu_{\mathfrak{m}}$ denote the product measure on the product of $\mathfrak{m}$ copies of $[0, 1]$ (using Lebesgue measure on each factor), we thus have that the spaces $L^\infty(\mu_{\mathfrak{m}})$ form a complete set of isomorphism types for the spaces $L^\infty(\mu)$ for arbitrary μ. Previous to our work, the spaces $L^p(\mu)$ for $1\leqq p<\infty$, $p\neq 2$ had been classified by Joram Lindenstrass as follows: if $\mathfrak{m}=\dim L^1(\mu)$ and $\mathfrak{m}>\aleph_0$, then if $\mathfrak{m}$ is not the limit of a (denumerable) sequence of smaller cardinals, $L^p(\mu)$ is isomorphic to $L^p(\mu_{\mathfrak{m}})$; if $\mathfrak{m}$ is such a limit, there are two mutually exclusive alternatives:

(1) $L^p(\mu)$ is isomorphic to $L^p(\mu_{\mathfrak{m}})$;

(2) choosing a fixed sequence of cardinals $\mathfrak{n}_1, \mathfrak{n}_2, \cdots$ with $\mathfrak{n}_k\to\mathfrak{m}$ and $\mathfrak{n}_k<\mathfrak{m}$ all k, $L^p(\mu)$ is isomorphic to $(L^p(\mu_{\mathfrak{n}_1})\oplus L^p(\mu_{\mathfrak{n}_2})\oplus\cdots)_p$. (If $\mathfrak{m}=\aleph_0$, it is a known result that $L^p(\mu)$ is isomorphic either to $L^p[0, 1]$ or to l^p, and these possibilities are mutually exclusive.)

The next result is considerably stronger than Theorem 3.2.

THEOREM 3.3. *Let A be a closed subspace of $L^1(\mu)$, and let* $\mathfrak{m}=\dim A$.

(a) *If B is a Banach space with B^* isomorphic to A^*, B is isomorphic to a subspace of $L^1(\mu_{\mathfrak{m}})$ and* $\dim B=\mathfrak{m}$. *If B is isomorphic to a subspace of $L^1(\nu)$ for some ν and* $\dim B<\mathfrak{m}$, *then there exists no one-to-one bounded linear operator from A^* to B^*.*

(b) *Suppose that A^* is injective. Then A^* is isomorphic to a subspace of $L^\infty(\mu_{\mathfrak{m}})$, and A^* is not isomorphic to a double conjugate Banach space unless $\mathfrak{m}=\aleph_0$, in which case A^* is isomorphic to l^∞.*

(c) *If A^* is isomorphic to $L^\infty(\nu)$ for some ν, then $L^1(\mu_{\mathfrak{m}})$ is isomorphic to a quotient space of A.*

This is proved by applying many of the previous results for parts (a) and (b), and the techniques of the proof of Theorem 3.1 for part (c).

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