OCTONION PLANES IN CHARACTERISTIC TWO

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1. Roughly speaking an octonion plane \(\mathcal{O}(\mathcal{O})\) is a plane coordinatized by an octonion (Cayley-Dickson) algebra \(\mathcal{O}\). The most successful approach to these planes has been made by first constructing a reduced exceptional simple Jordan algebra \(\mathfrak{J} = \mathfrak{J}(\mathcal{O}, \gamma)\) and then using \(\mathfrak{J}\) to define \(\mathcal{O}(\mathcal{O})\). The results on \(\mathcal{O}(\mathcal{O})\) in \([2],[7],[8],[10]\) and \([13]\) for \(\mathcal{O}\) an octonion division algebra and those in \([9],[11]\) and \([12]\) for split \(\mathcal{O}\) were obtained along these lines. However, in all of these papers the characteristic of the field is not two. In the present paper, we give a definition of octonion planes based on quadratic Jordan algebras. This definition is valid for all characteristics and both types of octonion algebras. We also indicate how most of the results mentioned above can be derived in this general setting.

We refer the reader to McCrimmon \([4]\) for a definition of a quadratic Jordan algebra. Recall also that the set \(\mathfrak{J}(\mathcal{O}, \gamma)\) of 3 by 3 matrices with entries in \(\mathcal{O}\) which are symmetric with respect to the involution \(x \mapsto \gamma^{-1}x^\gamma\) (where \(\gamma = \text{diag}\{\gamma_1, \gamma_2, \gamma_3\}, 0 \neq \gamma_i \in \Phi\), the field) can be endowed with the structure of a quadratic Jordan algebra. (See \([5]\).) We shall use the notation of \([4]\) and \([5]\) with the exception that we write our operators on the right and define \(zV_{x,y} = \{zxy\} = xU_{x,y}\). We remark that \(U_{x,y} = U_{x+y} - U_x - U_y\) where in characteristic not two \(U_x = 2R_x^2 = R_x\) where \(yR_x = \frac{1}{2}(xy+yx)\). Also \(x^T\) in \([5]\) is roughly the adjoint of the matrix \(x\) and \(x^T y = (x+y)^T - x^T - y^T\).

2. Let \(\mathfrak{J} = \mathfrak{J}(\mathcal{O}, \gamma)\) and let \(x_\ast\) and \(x^\ast\) be two copies of \(\{ax| 0 \neq a \in \Phi\}\) where \(x \in \mathfrak{J}\) is of rank one; i.e., \(x \neq 0\) but \(x^T = 0\). We define the octonion plane \(\mathcal{O}(\mathfrak{J})\) to have points \(x_\ast\) and lines \(y^\ast\), where \(x\) and \(y\) are of rank one, and relations (cf. \([9]\))

1. \(x_\ast y^\ast\), \(x_\ast\) incident to \(y^\ast\), if \(V_{x,y} = 0\),
2. \(x_\ast \sim y^\ast\), \(x_\ast\) connected to \(y^\ast\), if \(T(y, x) = 0\),
3. \(x_\ast \sim y_\ast\), \(x_\ast\) connected to \(y_\ast\), if \(y \times x = 0\),
4. \(x^\ast \sim y^\ast\), \(x^\ast\) connected to \(y^\ast\), if \(y \times x = 0\).

1 These results are contained in the author's doctoral dissertation written under the guidance of Professor N. Jacobson at Yale University. A more detailed paper is forthcoming. The author was a National Science Foundation Graduate Fellow while at Yale.
The groups $\Gamma$, $G$, and $S$ of semisimilarities, similarities, and norm preserving transformations respectively of $\mathfrak{J}$ can be defined as usual. If $W \in \Gamma$ has multiplier $\rho$ and associated isomorphism $s$, then $(xW)\gamma = \rho(xW)$ and $W^{-1} V_{x,y} W = V_{x\hat{w},y\hat{w}}$ for $x, y \in \mathfrak{J}$ where $\hat{W} = W^{*} = 1$ and $T(xW*, y) = T(x, yW)$. In particular, $W$ permutes the elements of rank one and the map $[W^1]: xW \mapsto (xW)*$ preserves the relations (1)-(4). Thus, $W \mapsto [W^1]$ defines a homomorphism of $\Gamma$ into the collineation group of $\varphi(\mathfrak{J})$. Let $\varphi(\mathfrak{J})$, $\varphi(\mathfrak{J})$, $\varphi(\mathfrak{J})$ respectively be the images of $\Gamma$, $G$, $S$ respectively. The fundamental theorem of octonion planes says that $\varphi(\mathfrak{J})$ is the entire collineation group or, more generally, that any collineation of two octonion planes is induced by a semisimilarity of the quadratic Jordan algebras. Also, since $\mathfrak{J}(\mathfrak{J}, \gamma)$ and $\mathfrak{J}(\mathfrak{J}, \gamma')$ are norm similar, we see that $\varphi(\mathfrak{J})$ depends only on $\mathfrak{J}$, and we write $\varphi(\mathfrak{J}) = \varphi(\mathfrak{J})$.

The points $u*, v*$, $w*$ from a three-point if $T(u, v \times w) \neq 0$. Four points form a four-point if each subset of three points is a three-point. A lemma (due to Ferrar for (linear) Jordan algebras) is that if $u*, v*, w*$ are a three-point then $u, v, w$ are pairwise orthogonal primitive idempotents in some isotope of $\mathfrak{J}$. This can be used to show

**Lemma 1.** $PS$ is transitive on three-points while $PG$ is transitive on four-points.

By means of Lemma 1, many of the geometric questions about $\varphi(\mathfrak{J})$ can be reduced to consideration of special points and lines. One can show $u*, v*$ are incident to exactly one line if and only if $u* \parallel v*$. Also, one sees that $u* \cong v*$ if and only if there exists $w* \parallel v*$ with $u* \cong w*$. In this case, if $x* \parallel v*$ then either $u* \cong x*$ or the unique line through $u*$ and $x*$ is connected to $v*$.

3. An important result on octonion planes is

**Theorem 1.** $PS$ is a simple group.

The proof of Theorem 1 is based on [1, Lemma 4, p. 39]. By Lemma 1 and other transitivity properties of $PS$, one can show that $PS$ is a primitive permutation group of the lines of $\varphi(\mathfrak{J})$. If $u \in \mathfrak{J}$ is of rank one and $v \in \mathfrak{J}$ with $T(u, v) = 0$, then the map $T_{u,*} = 1 + V_{u,*} + U_{u}U_{*}$ is a norm preserving map and $[T_{u,*}]$ fixes $u*$. $T_{u,*}$ is called an *algebraic transvection* and $[T_{u,*}]$ is called a *transvection*. The group $H_{u,*}$ generated by $[T_{u,*}]$ for a fixed $u$ is a normal abelian subgroup of the subgroup of $PS$ fixing $u*$. The other conditions of [1, Lemma 4, p. 39], namely that the $H_{u,*}$'s generate $PS$ and that $PS$ be its own derived group follow from
THEOREM 2. \( S \) is generated by algebraic transvections.

The main step in the proof of Theorem 2 is the identification of a certain subgroup of \( S \) with a spin group. Let \( e \) be a primitive idempotent in \( \mathcal{Z} \) and let \( \mathcal{Z}_0 \) be the Peirce 0-space of \( \mathcal{Z} \) relative to \( e \). If \( x \in \mathcal{Z}_0 \), then let \( \tilde{x} = e \times x \in \mathcal{Z}_0 \). Also, one has the quadratic form \( Q \) on \( \mathcal{Z}_0 \) where \( Q(x) = T(x^2) \). We can now state

**THEOREM 3.** Let \( u \in \text{Spin}(\mathcal{Z}_0, Q) \) and write \( u = v_1v_2 \cdots v_{2r} \) with \( v_i \in \mathcal{Z}_0 \). If \( W_u = U_{\theta(e_1)} U_{\theta(e_2)} \cdots U_{\theta(e_{2r-1})} U_{\theta(e_r)} \) where \( \theta(v) = e + v \), then \( u \to W_u \) is an isomorphism of \( \text{Spin}(\mathcal{Z}_0, Q) \) onto \( H \), the group of norm preserving maps fixing \( e \) and stabilizing \( \mathcal{Z}_0 \).

A corollary of Theorem 3 is that the group of automorphisms of \( \mathcal{Z} \) fixing \( e \) is isomorphic to \( \text{Spin}(\mathcal{Z}_0, Q, 1 - c) \) where in general \( \text{Spin}(V, Q, c) \) is defined in terms of the Clifford algebra \( C(V, Q, c) \) with basepoint much as \( \text{Spin}(V, Q) \) is defined in terms of the the Clifford algebra \( C(V, Q) \). \( C(V, Q, c) \), which has been studied by Jacobson and McRimmon in some unpublished notes, is the tensor algebra on \( V \) modulo the ideal generated by \( x \otimes x - T(x)x + Q(x)1 \) and \( c - 1 \) where \( T(x) = Q(x, c) \) and \( Q(c) = 1 \).

4. \( \mathcal{P}(\mathcal{O}) \) is a projective plane if and only if \( \mathcal{O} \) is a division algebra. In this case, there are two kinds of involutions in \( \mathcal{P}G \); those which fix a line of points and those which fix a four-point. In characteristic two, those of the first kind are transvections while those of the second kind are conjugate (under \( \mathcal{P}S \)) to an involution induced by an automorphism of \( \mathcal{Z} \) which is given by applying an automorphism (of order two) of \( \mathcal{O} \) to each entry of the 3 by 3 matrix. A useful result is

**LEMMA 2.** \( J \) is an automorphism of order two of \( \mathcal{O} \) in characteristic two if and only if one of the following holds:

(a) \( \mathcal{O} = \mathcal{B} \oplus \mathcal{B}_s \), where \( \mathcal{B} \) is a totally isotropic subalgebra of \( \mathcal{O} \) relative to the norm \( n \) on \( \mathcal{O} \), \( n(1, s) = 1 \), and \( J : b_1 + b_2s \mapsto b_1 + b_2s, b_i \in \mathcal{B} \).

(b) \( \mathcal{O} = \mathcal{F}_3 \oplus x \mathcal{F}_2 \) where \( n(x) = 1 \) and multiplication is given by [6, p. 45], and \( J \) corresponds to

\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}.
\]

Lemma 2 is used to show that any isomorphism of groups \( \mathcal{P}G \) or \( \mathcal{P}S \) of two projective planes in characteristic two preserves the type of an involution. This, in turn, is used to show that any such isomorphism is induced by a collineation or correlation of the planes.
5. We call \( \pi_0 : x^* \rightarrow x^* \) the standard polarity on \( \mathcal{S} \). We consider the group \( PT(\pi_0) \) generated by transvections commuting with \( \pi_0 \). In characteristic two, by using methods similar to those in [13] and [12], one can show that the group \( PT(\pi_0) \) is simple and if \( \Phi \) has more than two elements then \( \text{Aut} \mathfrak{S} \equiv PT(\pi_0) \). The simplicity of \( \text{Aut} \mathfrak{S} \) was shown in [3] in the setting of restricted Lie algebras.

References


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