MANIFOLDS OF THE HOMOTOPY TYPE OF 
(NON-LIE) GROUPS
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Communicated by Edgar Brown, February 17, 1969

Hilbert's Fifth Problem implies that a topological group which is
topologically a finite dimensional manifold is a Lie group. Until
quite recently, the only topological groups of the homotopy type of
compact manifolds known were Lie groups. In 1963 Slifker exhibited
a topological group of the homotopy type of $S^3$ yet not multiplica-
tively equivalent to $SU(1)$. In 1968, Hilton and Roitberg announced
the discovery of a 10-dimensional manifold $M_7^{10}$ which admits a
multiplication yet is not of the homotopy type of a Lie group. In
fact, they showed $M_7^{10} \times S^9 = \text{Sp}(2) \times S^9$. They left open the question:
Does $M_7^{10}$ admit a homotopy associative multiplication, a necessary
condition for $M_7^{10}$ to be of the homotopy type of a topological group?
We answer the question affirmatively; thus a homotopy version of
Hilbert's Fifth Problem is false.

**Theorem 1.** There is a topological group $G$ of the homotopy type of a
compact manifold $M_7^{10}$ (the 3-sphere bundle over $S^7$ described by Hilton
and Roitberg) which is not of the homotopy type of any Lie group.

More precisely we show the following

**Theorem 2.** Let $S^7 \to M_7^{10} \to S^7$ be the principal $S^7$-bundle classified
by $n\omega \in \pi_8(S^7)$, $n \in \mathbb{Z}_{12}$, $\omega$ chosen as a generator such that the correspond-
ing $M_7^{10}$ is $\text{Sp}(2)$.

- $M_7^{10}$ is of the homotopy type of a Lie group if and only if $n = \pm 1$ (12).
- $M_7^{10}$ is of the homotopy type of a topological group if $n = \pm 1, \pm 5$ (12).
- $M_7^{10}$ admits a multiplication if $n \neq 2$ (4).

The first part results from the classification of such bundles up
to homotopy type and the classification of Lie groups. The case
$n = -1$ is realized by $\text{Sp}(2)$, the opposite symplectic group, which
has the same underlying space as $\text{Sp}(2)$ but the opposite order of
multiplication.

The remainder of the theorem is proved using a new technique of
Zabrodsky's called "mixing homotopy types" [2].

Let $P$ be the set of primes and $P = P_1 \cup P_2$, a decomposition into
disjoint subsets. Let $\mathcal{C}P_1$ denote the class of abelian groups of orders
not divisible by primes in $P_2$ and let $\mathcal{C}P_2$ denote the class of abelian
groups not divisible by primes in $P_1$.

Let $X$, $X_0$ be simply connected CW-complexes.
Theorem 3. Let $f : X \to X_0$ be a rational homotopy equivalence. There exists a space $X(P_i)$ and a factorization $X \to X(P_i) \to X_0$ of $f$ such that the fibre of $f_i$ has homotopy groups belonging to $\mathbb{C}P_i$.

If $X$, $X_0$ are H-spaces and $f$ an H-map, then $X(P_i)$ is an H-space and $f_2$, $f_1$ are H-maps.

Theorem 4. Let $X_i$ be simply connected CW-complexes for $i = 0, 1, 2$. Let $f_i : X_i \to X_0$ be a rational homology equivalence.

There exists a space $X$ and maps $g_i : X \to X_i(P_i)$ such that the fibre has homotopy groups belonging to $\mathbb{C}P_i \pm 1$. If the “ingredients” $X_i$, $f_i$ are H-spaces and H-maps, then $X$ is an H-space and the maps $g_i$ are H-maps.

If the ingredients are topological groups and homomorphisms, $X$ has the homotopy type of a topological group.

Theorem 3 can be proved by constructing a modified Moore-Postnikov system for $f$ in which the primary components of the homotopy groups $\pi_i(X_0, X)$ are put in first for $p \in P_i$ and then for $p \in P_2$. More simply, since $f$ is a rational equivalence, its fibre $F$ has the homotopy type of a product $\prod_{p \in P_i} F_p$ where $F_p$ has $p$-primary homotopy only. $X(P_i)$ can be thought of as the subfibration of $X$ in which the fibre is cut down to $\prod_{p \in P_i} F_p$.

To obtain the $H$-conditions, the following specific details are helpful. $X(P_i)$ can be constructed by a succession of principal $K(\mathbb{Z}_p, n)$-fibrations, $p \in P_i$ induced by cohomology classes in the kernel of the cohomology morphism mod $p$.

Lemma. Let $f : X \to X^1$ be an H-map. If $f^* : H^i(X^1, \mathbb{Z}_p) \to H^i(X, \mathbb{Z}_p)$ is an isomorphism for $i < n - 1$, monomorphism for $i = n - 1$, and $\alpha \in \text{Ker} f^* \cap H^*(X^1, \mathbb{Z}_p)$ then $\alpha$ is represented by an H-map. The fibration $Y$ induced over $X^1$ by $\alpha$ is therefore an H-space such that $f$ can be lifted to an H-map $X \to Y$.

That $\alpha$ is represented by an H-map follows from the fact that $f^* \alpha = 0$ is represented by an H-map and the obstructions lie where $(f \times f)^*$ is an isomorphism. The vanishing of these obstructions can thus be achieved in terms of chains whose images in $X^2$ are specified so the lifting of $f$ is immediate.

Elements in the cokernel of the cohomology morphism are added by trivial principal $(\mathbb{Z}_p, m)$-fibrations.

Theorem 4 is proved by taking $X$ to be the fibre product (pull back) of $f_i : X_i(P_i) \to X_0$.
X \to X_1(P_1) \\
\downarrow \quad \downarrow \\
X_2(P_2) \to X_0

If the ingredients are topological groups and homomorphisms, Browder observed the construction can also be carried out in terms of $BX_i$ to produce a classifying space $Y$. We then have $X$ of the homotopy type of $\Omega Y$ and hence of the homotopy type of a topological group by Milnor's constructions, if all spaces are countable CW.

**Proposition.** If $X_1, X_2$ are simply connected finite complexes, then $X$ has the homotopy type of a finite complex.

**Proof.** Since $H^*(X_i; Q)$ is finite dimensional as a $Q$-vector space so is $H^*(X; Q)$. Since $H^*(X_i; Z_p)$ is finite dimensional for each $p, i=1, 2$, so is $H^*(X; Z_p)$. Moreover, the finite dimension has a common finite upper bound for $Q$ and all $p$ simultaneously (i.e. the maximum for $X_1, X_2$). Thus $X$ has the homotopy type of a finite complex,

We are now ready for examples. Let $\overline{Sp}(2)$ denote the "opposite symplectic group," i.e. the symplectic group obtained by multiplying quaternions in the opposite order. If $\omega \in \pi(S^3)$ is chosen as the generator which classifies $S^3 \to Sp(2) \to S^7$, then $\overline{Sp}(2)$ is classified by $-\omega$. Recall that $\pi_6(S^3) \approx \mathbb{Z}_4 \times \mathbb{Z}_2$ with generators $\nu ', \alpha$ [1]. We have $\omega = \nu ' + \alpha$. If we mix the homotopy type of $X_1 = Sp(2)$ and $X_2 = \overline{Sp}(2)$ over $K(\mathbb{Z}, 3) \times K(\mathbb{Z}, 7)$, the resulting group is the Hilton-Roitberg example, for $-\nu '+\alpha = 7\omega$. Interchanging $Sp(2)$ and $\overline{Sp}(2)$ gives a group classified as a bundle by $5\omega$.

If we take $X_1 = Sp(2)$ and $X_2 = S^3 \times S^7$ with $2 \in P_1$ and $3 \in P_3$ then the bundle classified by $\nu ' = 9\omega$ is seen to admit a multiplication but not a homotopy associative one [2]: $H^3 \to H^7$ is trivial which contradicts the nontriviality of cup cubes in the projective 3-spaces for $X$. The same holds for $3\omega$.

If we interchange the roles of $Sp(2)$ and $S^3 \times S^7$ then we see the bundles classified by $\pm 4\omega = \pm \alpha$ admit multiplications. The multiplication may be homotopy associative, although $S^7(\{2\})$ admits no homotopy associative multiplication.

**References**


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