ANALYTIC SHEAVES OF LOCAL COHOMOLOGY

BY YUM-TONG SIU

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Suppose $\mathcal{F}$ is a coherent analytic sheaf on a complex analytic space $X$. Denote by $S_k(\mathcal{F})$ the analytic subvariety $\{x \in X | \text{codh} \mathcal{F}_x \leq k\}$. For any open subset $D$ of $X$, denote by $S_k(\mathcal{F}|D)$ the topological closure of $S_k(\mathcal{F}|D)$ in $X$. If $V$ is an analytic subvariety of $X$, denote by $\mathcal{K}_V^k(\mathcal{F})$ the sheaf defined by the presheaf $U \mapsto H^k_U(U, \mathcal{F})$, where $H^k_U(U, \mathcal{F})$ is the $k$-dimensional cohomology group of $U$ with coefficients in $\mathcal{F}$ and supports in $V$. If $\phi: X \to Y$ is a holomorphic map, denote by $\phi^*(\mathcal{F})$ the $k$th direct image of $\mathcal{F}$ under $\phi$. If $X, \mathcal{F}$, and $V$ are complex algebraic instead of analytic, $\mathcal{K}_V^k(\mathcal{F})$ has the same meaning and $\mathcal{F}^h$ denotes the coherent analytic sheaf canonically associated with $\mathcal{F}$.

Our results are as follows:

**Theorem A.** Suppose $V$ is an analytic subvariety of a complex analytic space $(X, \mathcal{K})$, $q$ is a nonnegative integer, and $\mathcal{F}$ is a coherent analytic sheaf on $X$. Let $\theta: X - V \to X$ be the inclusion map. Then the following three statements are equivalent:

(i) $\theta_0(\mathcal{F}|X-V), \cdots, \theta_q(\mathcal{F}|X-V)$ (or equivalently $\mathcal{K}^0_V(\mathcal{F}), \cdots, \mathcal{K}^{q+1}_V(\mathcal{F})$) are coherent on $X$.

(ii) For every $x \in V$, $\theta_0(\mathcal{F}|X-V)_x, \cdots, \theta_q(\mathcal{F}|X-V)_x$ (or equivalently $\mathcal{K}^0_V(\mathcal{F})_x, \cdots, \mathcal{K}^{q+1}_V(\mathcal{F})_x$) are finitely generated over $\mathcal{O}_x$.

(iii) $\dim V \cap \overline{S_{k+q+1}(\mathcal{F}|X-V)} < k$ for every $k \geq 0$.

**Theorem B.** Suppose $V$ is an algebraic subvariety of a complex algebraic space $X$, $q$ is a nonnegative integer, and $\mathcal{F}$ is a coherent algebraic sheaf on $X$. Then $\mathcal{K}^0_V(\mathcal{F}), \cdots, \mathcal{K}^{q+1}_V(\mathcal{F})$ are coherent algebraic sheaves on $X$ if and only if $\mathcal{K}^0_V(\mathcal{F}^h), \cdots, \mathcal{K}^{q+1}_V(\mathcal{F}^h)$ are coherent analytic sheaves on $X$. If so, the canonical homomorphisms $\mathcal{K}^k_V(\mathcal{F}^h) \to \mathcal{K}^k_V(\mathcal{F}^h)$ are isomorphisms for $0 \leq k \leq q+1$.

In the theory of extending coherent analytic sheaves, the main problem is to answer the following question: Suppose $\mathcal{F}$ is a coherent analytic sheaf on $X-V$, where $V$ is an analytic subvariety of a complex analytic space $X$. Let $\theta: X-V \to X$ be the inclusion map. When is $\theta_0(\mathcal{F})$ coherent? This question has been answered in various ways in [1] through [10]. Theorem A gives a criterion for the coherence of $\theta_q(\mathcal{F})$ after a coherent analytic extension has been found. This criterion given in Theorem A sharpens a result of Trautmann [11]. Theorem B answers in the affirmative a question raised by Serre [2, pp. 373–374].

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The proofs consist of refining the techniques in [11] and skillfully making use of results concerning gap-sheaves and homological co-dimensions and zero-divisors of stalks of coherent sheaves. There is an algebraic analog of the same formulation for Theorem A. Details will appear elsewhere.

ADDED IN PROOF. In a paper to be published, Trautmann independently has also obtained the equivalence of (i) and (iii) of Theorem A.

REFERENCES


UNIVERSITY OF NOTRE DAME, NOTRE DAME, INDIANA 46556