ON ABSOLUTELY CONTINUOUS TRANSFORMATIONS

BY JAMES K. BROOKS AND PAUL V. REICHELDERFER

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1. Introduction. In this announcement, we examine absolutely continuous transformations $T$ mapping the measure space $(S, \Sigma, \mu)$ onto the measure space $(S', \Sigma', \mu')$. In order to obtain information about the change of measure induced by $T$, a weight function $W'$ defined on $S' \times \mathfrak{Q}$ is introduced, where $\mathfrak{Q}$ is a certain subfamily of $\Sigma$. $W'(s', D)$ represents a weight assigned to the points in $D$ which $T$ maps into $s' \in S'$. We present structure theorems (Theorems 2 and 3) for weight functions which enable us to establish a transformation formula (Theorem 1) for integrals defined on the measure spaces. Theorem 1 includes all the existing transformation formulas for transformations which are absolutely continuous with respect to a real valued weight function. Moreover, the integrability condition necessary to ensure the existence of the formula is minimal, as we shall indicate in §3.

Rado and Reichelderfer [11] considered the case when the measure spaces are Euclidean $n$-space (both having the same dimension), with Lebesgue measurable sets and $n$-dimensional Lebesgue measure; $T$ is a bounded continuous transformation defined on the bounded domain $S$. In particular, the weight function $\mu_*(s', T, D)$ generated by the topological index defined on indicator domains is used to define an essentially absolutely continuous transformation. Also the Banach indicatrix or crude multiplicity function $N(s', T, D)$ and the weight function $k(s', T, D)$ which counts the number of essential maximal model continua for $(s', T, D)$ are treated in detail in [11]. In this classical setting, Craft [10] removed some conditions on the weight functions. Reichelderfer [13] developed a transformation theory for general measure spaces under certain standard hypotheses. Necessary and sufficient conditions were given in order that a transformation be absolutely continuous. In [14] it was shown that a large class of topological spaces satisfies these hypotheses; consequently, in this general topological setting the concepts of absolute continuity and generalized Jacobians can be effectively defined. Brooks [1], [3] developed the theory for integrals in Banach spaces and introduced a larger class of weight functions [2]; as a special case, signed weight functions may now be used when the spaces are oriented. Lebesgue decomposition theorems and measurability theorems for positive weight functions were considered by Chaney [6], [7], [8].
The theorems in §3 will be used at a later date to develop a chain rule for the product of absolutely continuous transformations. This problem was first solved for Euclidean $n$-space in [12] and was treated in a more general setting in [9].

2. The setting. In this section we establish notation and state some definitions. Throughout this paper we assume that the standard hypotheses for transformation theory H1–H8 are satisfied (see [13] or [1]). For the reader’s convenience some of the families of sets occurring in these hypotheses are listed.

$(S, \Sigma, \mu)$ and $(S', \Sigma', \mu')$ are $\sigma$-finite complete measure spaces. $T$ is a function (transformation) mapping $S$ onto $S'$. $\mathcal{D}$ is a subfamily of $\Sigma$ containing $\emptyset$ and $S$. $D$ will be a generic notation for a set in $\mathcal{D}$. $T\mathcal{D} \subseteq \Sigma'$ and the intersection of two sets in $\mathcal{D}$ can be written as a countable union of disjoint sets from $\mathcal{D}$. For every $E \in \Sigma$ and $\epsilon > 0$ there exists a disjoint sequence $\{D_i\}$ such that $E \subseteq \bigcup D_i$ and $\mu(\bigcup D_i - E) < \epsilon$. A set $E \subseteq S$ is $\mu'$-null if $E = A_1 \cup A_2$, where $\mu(A_1) = \mu'(TA_2) = 0$. $\Delta(D)$ ($\Delta^*(D)$) denotes the family of all finite (countable) collections of pairwise disjoint sets in $\mathcal{D}$ contained in $D$. $\mathbb{R}$ denotes the real numbers.

A signed weight function for $T$ is a real valued function $W'$ defined on $S' \times \mathcal{D}$ such that:

(i) $W'(\cdot, D) = 0$ a.e. $\mu'$ on $S' - TD$;

(ii) If $D_i \uparrow D$, then $\lim W'(\cdot, D_i) = W'(\cdot, D)$ a.e. $\mu'$;

(iii) If $\{D_i\} \in \Delta^*(D)$ and $D - \bigcup D_i$ is $\mu'$-null, then $W'(\cdot, D) = \sum W'(\cdot, D_i)$ a.e. $\mu'$;

(iv) $W'(\cdot, D)$ is measurable for each $D$.

$W'$ will always denote a signed weight function.

A positive weight function $G'$ is a nonnegative function satisfying the above conditions except (iii) is replaced by the requirement that $\sum G'(\cdot, D_i) \leq G'(\cdot, D)$ a.e. $\mu'$, whenever $\{D_i\} \in \Delta^*(D)$. Write $G_i' \ll G_2'$ if $G_i' (\cdot, D) \leq G_2' (\cdot, D)$ a.e. $\mu'$ for every $D$.

$T$ is of bounded variation with respect to $W'(BVW')$ if there exists a nonnegative function $K' \in L_1(\mu')$ such that for $\{D_i\} \in \Delta^*(S)$,

$\sum |W'(\cdot, D_i)| \leq K'$ a.e. $\mu'$. $T$ is absolutely continuous with respect to $W'$ ($ACW'$) if $T$ is $BVW'$ and there exists a function $f \in L_1(\mu)$ such that $\int_{D_i} \mu' = \int_{TD} W'(\cdot, D) d\mu'$, for every $D$. $f$ is called a generalized Jacobian for $T$ relative to $W'$ (it follows that $f$ is unique in $L_1(\mu)$).

A function $g$ satisfies condition $(N)_T$ if $g = 0$ a.e. $\mu$ on $T^{-1}E'$ whenever $\mu'(E') = 0$. Let $W(D) = \int_{TD} W'(\cdot, D) d\mu'$, $D \in \mathcal{D}$. Define

$$V(D, W) = \sup_{\Delta(D)} \sum |W(D_i)|; V'(\cdot, D) = \sup_{\Delta(D)} \sum |W'(\cdot, D_i)|.$$
The above definitions include the existing definitions of bounded variation and absolute continuity in the literature. We assume in the sequel that $T$ is $ACW'$ with generalized Jacobian $f$.

3. The results. Proofs for the theorems in this section will appear elsewhere. The main result is the following

**Theorem 1 (Transformation Formula).** Let $H': S' \to \mathbb{R}$ be measurable. Then $H' \circ Tf$ is measurable. If $H' \circ Tf$ is $\mu$-integrable on a fixed set $D$, then $H'W'(\cdot, D)$ is $\mu'$-integrable and

$$
\int_D H' \circ Tfd\mu = \int_D H'W'(\cdot, D)d\mu'.
$$

As mentioned above, this theorem extends the results of Rado and Reichelderfer in Euclidean $n$-space [11, p. 262]. We mention that the integrability of $H'W'(\cdot, D)$ does not imply the integrability of $H' \circ Tf$ on $D$ [4]; however, the integrability of $H'V'(\cdot, D)$ implies the integrability of $H' \circ Tf$ on $D$ [2]. The following results which are used to establish Theorem 1 are interesting in their own right.

**Theorem 2.** $f$ satisfies condition $(N)_T$ and $V(D, W) = \int_D |f|d\mu$.

The proof of the first assertion involves a technique similar to the one found in the proof of Lemma 3.1 in [2]. The proof of the second part is a long technical argument using $\gamma$-type partitions of elements of $\mathcal{D}$ [13].

The next theorem yields a decomposition for weight functions which complements the pointwise decomposition theorems presented in [5].

**Theorem 3 (Jordan Decomposition).** There exist positive weight functions $Q', Q'_{\pm}$ for $T$ such that

1. $Q' = Q'_{++} + Q'_{--}; \quad W' = Q'_{++} - Q'_{--}$.
2. $T$ is $ACQ'$, $ACQ'_{\pm}; \quad |f|(f^{\pm})$ is generalized a Jacobian for $T$ relative to $Q'$ ($Q'_{\pm}$).
3. If $U', U_{\pm}'$ are nonnegative real valued functions defined on $S' \times \mathcal{D}$ such that $W' = U'_{++} - U'_{--}$ and $U' = U'_{++} + U'_{--}$ is a positive weight function for which $T$ is $ACU'$, then $U_{\pm}'$ are positive weight functions, $T$ is $ACU'$, and $U_{\pm}' > Q'_{\pm}$.

In the proof, the function $\Psi_D(E') = \int_{(\sigma-U') \cap \mathcal{D}} |f|d\mu$, $E' \subseteq \Sigma'$ is used (cf. [8]). $\Psi_D$ is well defined on $\Sigma'$ and $\Psi_D \ll \mu'$ by Theorem 2. Hence, we can define $Q'(\cdot, D) = (d/d\mu')\Psi_D$. $Q'_{\pm}$ are defined by 1. By using
convergence theorems, one can show that $Q', Q_2'$ satisfy the conclusions of the above theorem.

BIBLIOGRAPHY


University of Florida, Gainesville, Florida 32601 and Ohio University, Athens, Ohio 45701