Let $M$ be an $F$-manifold, that is, a separable, metric manifold modelled on an infinite-dimensional Fréchet space. The question was raised at a problem seminar this January (1969) at Cornell University whether homotopic embeddings of another $F$-manifold in $M$ are isotopic. In this note the affirmative answer is given and a stronger result established.

Given an open cover $\mathcal{U}$ of a space $X$, two maps $f$ and $g$ of a space $Y$ into $X$ are said to be $\mathcal{U}$-close provided that for each $y$ in $Y$ there is an element of $\mathcal{U}$ containing both $f(y)$ and $g(y)$. The two maps are said to be pseudo-isotopic provided there is a map $h: Y \times I \to X$ with

$$h(y, 0) = f(y), \quad h(y, 1) = g(y)$$

and which for each $t$ in $(0, 1)$ is an embedding of $Y \times \{t\}$. The theorem is as follows:

**Theorem.** Homotopic maps of a separable metric space into an $F$-manifold are pseudo-isotopic. If the domain is complete, the pseudo-isotopy may be required to be through closed embeddings. Furthermore, given any open cover $\mathcal{U}$ of the manifold and any homotopy $F$ between the maps, the pseudo-isotopy may be required to be $\mathcal{U}$-close to $F$.

**Proof.** Let $X$ be the separable metric space, $M$ the $F$-manifold, and $f$ and $g$ the homotopic maps of $X$ into $M$. By a collection of results, all separable, infinite-dimensional Fréchet spaces are homeomorphic to the countably infinite product $s$ of open intervals $(-1, 1)$. (For a discussion of these results and a bibliography, see the introduction of [3].) Furthermore, a theorem of R. D. Anderson and R. M. Schori [4] asserts that given any open cover $\mathcal{U}$ of $M$, there is a homeomorphism $h_\mathcal{U}$ of $M$ onto $M \times s$ so that $\rho \circ h_\mathcal{U}$ is $\mathcal{U}$-close to the identity map, where $\rho$ is the projection onto $M$. If $\{s_t\}_{t=1}^\infty$ is a countable, indexed family of copies of $s$, it is easy to see that $s'$, the product of the $s_t'$s, is homeomorphic to $s$, so $s$ may be replaced by $s'$ in the above theorem.

For each integer $i$ and real number $t$ in $(-1, 1)$, let $\psi_{t,i}: s_i \to s_t$ be the map which multiplies in each coordinate by $t$, and let
\[ \phi(i, t) = 1 \quad \text{if } t \leq \frac{1}{i + 1} \text{ or } t \geq \frac{i}{i + 1}, \]
\[ = 0 \quad \text{if } \frac{1}{i} \leq t \leq \frac{i - 1}{i}, \]
\[ = (i + 1)(1 - it) \quad \text{if } \frac{1}{i + 1} \leq t \leq \frac{1}{i}, \]
\[ = (i + 1)(it - i + 1) \quad \text{if } \frac{i - 1}{i} \leq t \leq \frac{i}{i + 1}. \]

Also, let \( k_i \) be an embedding of \( X \) in \( s_i \), as a closed set if \( X \) is complete. (It is well known that this may be done in a separable Banach space.)

Given any homotopy \( F \) between \( f \) and \( g \) and any open cover \( \mathcal{U} \) of \( M \), let \( \mathcal{V} \) be a star-refinement of \( \mathcal{U} \); \( h_0 \), a homeomorphism of \( M \) onto \( M \times s' \) such that \( p \circ h_0 \) and the identity are \( \mathcal{V} \)-close, and define \( G: X \times I \to M \) by

\[ G(x, t) = h_0^{-1} \circ \left[ \text{id}_M \times \prod_{i=1}^{\infty} (\psi_{i, \phi(i, t)} + (\psi_{i, 1-\phi(i+1, t)} \circ k_i(x))) \right] \circ h_0 \circ F(x, t), \]

where "\( + \)" is understood to indicate coordinate-wise addition, and "\( \prod \)", the product of mappings.

For each \( t \) in \((0, 1)\), \( h_0 \circ G | X \times \{t\} \) may be regarded as the product of a mapping of \( X \) into \( M \times \prod_{i=1}^{\infty} s_i \) with a (closed) embedding of \( X \) in \( s_{i_0} \), where \( i_0 \) is any integer greater than or equal to both \( 1/t \) and \( 1/(1-t) \). It is a simple matter to see that this is a (closed) embedding since it is continuous, one-to-one, and the inverse is continuous because given a point \((x, t)\) and a sequence \( \{\alpha_i, t\}_{i=1}^{\infty} \) in \( X \times \{t\} \) for which \( h_0 \circ G(x_i, t) \) converges to \( h_0 \circ G(x, t) \), the \( s_{i_0} \)-coordinates of \( \{h_0 \circ G(x_i, t)\}_{i=1}^{\infty} \) converge to the \( s_{i_0} \)-coordinate of \( h \circ G(x, t) \), and as the mapping into the \( s_{i_0} \)-coordinate is an embedding, this forces \( \{x_i\}_{i=1}^{\infty} \) to converge to \( x \). The image \( h_0 \circ G(X \times \{t\}) \) is closed if \( X \) is complete, since if \( \{\alpha_i, t\}_{i=1}^{\infty} \) is a sequence in \( X \times \{t\} \) and \( p \) is in \( M \times s' \) with \( h_0 \circ G(x_i, t) \) converging to \( p \), then the \( s_{i_0} \)-coordinates of \( h_0 \circ G(x_i, t) \) converge to the \( s_{i_0} \)-coordinate of \( p \), which forces the \( s_{i_0} \)-coordinate of \( p \) to be \( k_{i_0}(x) \), for some \( x \), and thus forces \( \{x_i\}_{i=1}^{\infty} \) to converge to \( x \).

If it is desired, the pseudo-isotopy may be modified slightly to provide that
(a) it be an embedding of $X \times (0, 1)$ in $M$ and (b) the image of $X \times (0, 1)$ lie in a countable union of closed sets of $M$ each of which has Property Z in $M$ (in case $X$ is complete, the image of $X \times (0, 1)$ may be required to be the countable union of closed sets with Property Z in $M$).

(A closed set $Y$ has Property Z in $M$ provided that for each nonnull open set $U$ of $M$ with trivial homotopy groups, $U - Y$ be also nonnull and have trivial homotopy groups. The importance of Property Z for $F$-manifolds is demonstrated by [2] in which it is shown that the subsets of such which are homeomorphic to the manifolds by homeomorphisms $\mathcal{U}$-close to the identity for all open covers $\mathcal{U}$ are precisely the complements of countable unions of closed sets, each with Property Z.) The modified homotopy $H: X \times I \to M$ may be defined by setting $H(x, t) = h_0^{-1} \circ [id_M \times \prod_{i=1}^{\infty} \xi_i(x)] \circ h_0 \circ F(x, t)$, where

$$
\begin{align*}
\xi_i(x) &= \psi_{i,\theta(i+2,1)} + (\psi_{i,\theta(i+1,1)} \circ k_i(x)), \quad \text{if } i \text{ is even}, \\
&= \psi_{i,\theta(i+2,1)} + (\psi_{i,\theta(i+1,1)} \circ \psi_{i,\theta(y_i)}), \quad \text{if } i \text{ is odd} \\
&\quad \text{but not divisible by three, and} \\
&= \psi_{i,\theta(i+2,1)}, \quad \text{if } i \text{ is an odd multiple of three.}
\end{align*}
$$

Here, $y_i$ is merely a point in $s_i$ with not all coordinates zero; the $y_i$'s are introduced for the purpose of guaranteeing that $H \big| X \times (0, 1)$ be an embedding. The insertion of merely the $\psi_{i,\theta(i+2,1)}$ in infinitely many coordinates is to ensure that for any $t_0$ in $(0, 1/2)$, $h_0 \circ H(X) \times [t_0, 1-t_0]$ project into $s'$ on a set of infinite co-dimension which, by a theorem of R. D. Anderson [1], must have closure with Property Z. This guarantees that the closure of $h_0 \circ H(X) \times [t_0, 1-t_0]$ has Property Z in $M \times s'$ and hence that the closure of $H(X) \times [t_0, 1-t_0]$ has Property Z in $M$. If $X$ is complete, the construction gives that $H(X \times [t_0, 1-t_0])$ is closed and has Property Z for each $t_0$ in $(0, 1/2)$.

Remark. D. W. Henderson has recently proven in [5] that if $X$ is an $F$-manifold and $\mathcal{U}$ an open cover of $M$, than any map of $X$ into $M$ may be approximated $\mathcal{U}$-closely by closed and open embeddings.

In light of these results, the following question, also raised at Cornell, would seem to be the appropriate one: "Under which circumstances are two homotopic embeddings of one $F$-manifold in another ambient isotopic?"

**References**


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