1. Results. A recent result of H. S. Shapiro and A. L. Shields [4] states that if \( f \) and \( g \) are continuous complex valued functions on the unit interval \( I \) such that together they separate the points of \( I \) and also that \( f \) alone separates all but one pair of points, then the closed subalgebra of \( C(I) \) generated by \( f \) and \( g \) is all of \( C(I) \). Two generalizations are:

**Theorem.** Let \( A \) be a separating uniform algebra on \( I \) such that there exists an \( f \) in \( A \) which is locally 1-1, then \( A = C(I) \).

**Theorem.** Let \( A \) be a separating uniform algebra on \( I \) generated by two functions \( f \) and \( g \) such that there is a compact totally disconnected subset \( E \) of \( I \) such that

(i) \( f|E \) is constant, and
(ii) \( f \) separates every pair of points of \( I \) not both of which are in \( E \).
Then \( A = C(I) \).

The proofs use the notion of analytic structure in a maximal ideal space. J. Wermer first obtained results along these lines and further contributions were made by E. Bishop and H. Royden and then by G. Stolzenberg [5] who proved

**Stolzenberg’s Theorem.** Let \( X \subseteq \mathbb{C}^n \) be a polynomially convex set. Let \( K \subseteq \mathbb{C}^n \) be a finite union of \( C^1 \)-curves. Then \((X \cup K)^c - X \cup K\) is a (possibly empty) pure 1-dimensional analytic subset of \( \mathbb{C}^n - X \cup K \). (See [5] for the notation and definitions.)

A further result of Stolzenberg (and Bishop) is that a \( C^1 \) arc \( K \subseteq \mathbb{C}^n \) is polynomially convex and \( P(K) = C(K) \). It is well known that no smoothness is needed in \( C^1 \) but that in higher dimensions further assumptions are required for the above conclusion. We have

**Theorem.** Let \( f_1, f_2, \cdots, f_n \in C(I) \) separate the points of \( I \) and suppose that for \( 1 \leq i \leq n - 1 \), \( f_i \) is either \( C^1 \) or real-valued. Then the separating uniform algebra which \( f_1, f_2, \cdots, f_n \) generate is \( C(I) \).

If all the \( f_i, 1 \leq i \leq n - 1 \) are real valued, this theorem reduces to a result of Rudin [3]; on the other hand, if we consider the image \( K \) of \( I \) under \( t \to (f_1(t), \cdots, f_n(t)) \) we obtain a generalization of Stolzenberg’s result on smooth arcs.
Applied to uniform algebras on the circle \( T \), the methods of the previous theorems yield

**Theorem.** Let \( A \) be a separating uniform algebra on \( T \) which contains a function \( f \) which is locally 1-1, then either

(i) \( T \) is the maximal ideal space \( M_A \) of \( A \), in which case \( A = C(T) \) or

(ii) \( M_A - T \) is nonempty and has the structure of a 1-dimensional analytic space on which the functions in \( A \) are analytic.

Finally we have the following which Shapiro and Shields [4] conjectured as an improvement of a result of Björk.

**Theorem.** Let \( \Delta = \{ z : |z| < 1 \} \). Let \( F \) be a closed subset of \( \Delta \) with \( T \subseteq F \subseteq \Delta \) such that

(i) \( F \) has no interior in \( \mathbb{C} \),

(ii) \( \Delta - F \) is connected.

(iii) \( (\Delta \cap F)^c \) does not contain \( T \).

Let \( g \in C(F) \) and suppose that the separating uniform algebra on \( F \) generated by \( g \) and \( z \) is a proper subalgebra of \( C(F) \). Then there exists \( G \in C(\Delta) \) such that

(i) \( G | T = g | T \),

(ii) \( G \) is analytic on \( \Delta - F \).

The proofs [1] will appear elsewhere, together with more complete references to the literature. J. E. Björk [2] has independently obtained similar results.

2. A special case. In order to indicate the methods, we prove the following special case of the first mentioned theorem.

**Proposition 1.** Let \( A \) be a separating uniform algebra on \( I \) which contains a function \( f \) which separates all but a finite number of pairs of points of \( I \). Then \( A = C(I) \).

**Proof (Sketch).** It is easily seen that there are a finite number of functions in \( A \) which separate the points of \( I \) and so we may assume that \( A \) is finitely generated by \( f_1 = f, f_2, \cdots, f_n \). Let \( K \) be the homeomorphic image of \( I \) under the map \( t \rightarrow (f_1(t), \cdots, f_n(t)) \). Then \( K \) is an arc in \( \mathbb{C}^n \) and \( z_1 \) (the first coordinate function) separates all but a finite number of pairs of points of \( K \). Our goal is to prove \( P(K) = C(K) \). We note that \( C - z_1(K) \) has finitely many components and in order to give a proof by induction on this number we prove a more general result.

**Proposition 2.** Let \( K \) be a finite disjoint union of arcs in \( \mathbb{C}^n \). Suppose \( z_1 \) separates all but a finite number of pairs of points of \( K \). Then \( P(K) = C(K) \) and (hence) \( K \) is polynomially convex.
PROOF. Let \( L = \mathcal{C}_1(K) \). \( C^1 - L \) has finitely many components. The proof will be by induction on this number \( k \).

\( k = 1 \): \( L \) does not separate the plane and \( L \) has no interior and so \( P(L) = C(L) \). It follows that \( z \rightarrow z \) is in \( P(L) \) and so \( z \circ z_1 = z_1 \in P(K) \).

It is easily seen from the Stone-Weierstrass theorem that \( P(K) \) contains every \( f \in C(K) \) which identifies the points that \( z_1 \) does. From this it follows that \( P(K) = C(K) \).

Next we assume the result for \( k - 1 \) and prove it for \( k > 1 \). Assume, for the moment, that \( K \) has been proved to be polynomially convex. Then \( L = \mathcal{C}_1(K) \) is the spectrum of \( z_1 \) as an element of \( P(K) \). As \( R(L) = C(L) \) it follows from the Gelfand theory that \( F \circ z_1 \in P(K) \) for all \( F \in C(L) \). In particular, \( z_1 \in P(K) \) and, as above, \( P(K) = C(K) \).

It remains to show \( K \) is polynomially convex. Suppose not. Let \( \Omega \) be a bounded component of \( C^1 - L \) such that there is an arc \( \gamma \subseteq \partial \Omega \) which is also in the boundary of \( \Omega_\alpha \), the unbounded component of \( C^1 - L \). Let \( \gamma^0 \) denote \( \gamma \) with its endpoints deleted. We may assume \( z_1 \) is \( 1 \)-on \( z_1^{-1}(\gamma) \cap K \). Since \( \gamma \) is in the boundary of \( \Omega_\alpha \), \( z_1^{-1}(\gamma) \cap K = z_1^{-1}(\gamma) \cap \hat{K} \) by [5]. Let \( K_1 = K - z_1^{-1}(\gamma^0) \). Then \( K_1 \) satisfies the hypotheses of our proposition for the case \( k - 1 \). So by the induction hypothesis, \( P(K_1) = C(K_1) \) and \( K_1 \) is polynomially convex. We claim that \( z_1(\hat{K}) \cap \Omega \neq \emptyset \). In fact if \( p \in \hat{K} - K_1 \), as \( K_1 \) is polynomially convex, there is a polynomial \( f \) such that \( f(p) = 1 > \|f\|_{z_1} \). Let \( T \) be the component of \( \{q \in \hat{K} : |f(q)| \geq 1 \} \) which contains \( p \). Then by the local maximum modulus principle, \( T \) meets \( K \); hence \( T \) meets \( z_1^{-1}(\gamma^0) \cap K \). Hence \( z_1(T) \) meets \( \gamma \) and so clearly \( z_1(T) \) meets \( \Omega \).

Now by considering closed Jordan domains whose interiors are contained in \( \Omega \), whose boundaries contain \( \gamma \) and which meet \( \Omega \cap z_1(\hat{K}) \), it follows by [5] that \( z_1^{-1}(\Omega) \cap \hat{K} \) is a \( 1 \)-dimensional complex manifold in \( z_1^{-1}(\Omega) \) which is mapped by \( z_1 \) biholomorphically onto \( \Omega \).

Let \( \alpha \) be a straight line segment in \( \Omega \). Let \( \gamma_1 \) and \( \gamma_2 \) be arcs in \( \Omega \cap \{ \text{endpoints of } \gamma \} \) which join the endpoints of \( \alpha \) to those of \( \gamma \) such that \( \alpha \cap \gamma \cap \gamma_1 \cap \gamma_2 \) is a Jordan curve bounding an open Jordan domain \( W \subseteq \Omega \). Let \( J = z_1^{-1}(\alpha) \cap \hat{K} \). \( J \) is a real analytic arc in \( C^n \). Let \( X = (\hat{K} - z_1^{-1}(\gamma)) \cup (z_1^{-1}(\gamma_1 \cup \gamma_2) \cap \hat{K}) \). Then \( X \) is polynomially convex as it is a union of arcs such that \( C^1 - z_1(X) \) has \( k - 1 \) components. By Stolzenberg's theorem \( (X \cup J)^g - X \cup J \) is a \( 1 \)-dimensional analytic subset of \( C^n - X \cup J \). But by the local maximum modulus principle \( (X \cup J)^g = \hat{K} - z_1^{-1}(W \cup \gamma^0) \). It follows that \( \hat{K} - K \) is a \( 1 \)-dimensional analytic subset of \( C^n - K \). If \( \lambda \in \Omega \cap z_1(\hat{K}) \), then \( z_1 - \lambda \) is an analytic function on \( \hat{K} \) which has a zero on \( \hat{K} \) and has a logarithm on \( \hat{K} \); this contradicts the argument principle [5]. We conclude that \( \hat{K} = K \). Q.E.D.
REFERENCES


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