This note states results extending those of Nash [2] on isometric embeddings of Riemannian manifolds in euclidean spaces; proofs and further details will be given elsewhere.

Let $M$ be a $d$-dimensional $C^\infty$ manifold. For convenience, we assume throughout that manifolds, whether compact or not, are connected. A metric on $M$ is defined to be a quadratic form on the tangent bundle of $M$; note that there is no assumption of nondegeneracy. We shall assume that all metrics are $C^\infty$. A Riemannian metric on $M$ is a metric whose restriction to the tangent space $T_q$ at a point $q \in M$ is positive definite, for all $q \in M$. A pseudo-Riemannian, or indefinite, metric is a metric whose restriction to the tangent space at each point is nondegenerate; if the nondegenerate restriction to $T_q$ has $n$ negative eigenvalues and $p$ positive eigenvalues, with $p + n = d$, the metric is said to have signature $(p, n)$ at $q$. The connectedness of $M$ implies that the signature is independent of the choice of $q \in M$.

$\mathbb{R}^m$ will denote euclidean $m$-dimensional space, with the standard flat, positive definite metric, unless otherwise indicated; $\mathbb{R}^p_n$ denotes euclidean $(n+p)$-dimensional space with flat metric of signature $(p, n)$. Thus $\mathbb{R}^m_n = \mathbb{R}^m$. Let $F$ be a $C^\infty$ map, $F: M \to \mathbb{R}^p_n$, and let $g$ be a metric on $M$; $F$ is said to be isometric for $g$ if $F^*(\cdot) = g$ where $'^*$ denotes the metric for $\mathbb{R}^p_n$ indicated above. Note that if $g$ is Riemannian and $F$ is isometric for $g$, then $F$ is necessarily an immersion and $n + p \geq d = \text{dim } M$; for a general metric $g$, however, $F$ need not be an immersion. We shall concern ourselves with the question: given $M$ and a metric $g$ on $M$, for what $\mathbb{R}^p_n$ do there exist isometric immersions, or isometric embeddings, $F: M \to \mathbb{R}^p_n$?

1. A geometric argument for general metrics. Nash [2] guarantees the existence of isometric embeddings in some Riemannian euclidean space for any manifold with a Riemannian metric. The following argument reduces the general metric case to the Riemannian case, but requires higher dimension in the receiving euclidean space than necessary.

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**Proposition.** Suppose any Riemannian metric on $M$ has an isometric immersion in $\mathbb{R}^k$. Then any metric $g$ on $M$ has an isometric immersion (embedding) in $\mathbb{R}^k_{2d}(\mathbb{R}^{2d+1}_{2d+1})$.

**Outline of Proof.** Using Whitney's standard results on immersions and embeddings, one obtains a "large" immersion or embedding $E: M \to \mathbb{R}^{2d}$ or $\mathbb{R}^{2d+1}$ such that $E^*(\cdot) + g$ is Riemannian. Then, if $F$ is isometric for $E^*(\cdot) + g$, $F \times E: M \to \mathbb{R}^k_{2d}$ (or $\mathbb{R}^k_{2d+1}$) is isometric for $g$, where $F$ maps to the $k$ positive-eigenvalue coordinates and $E$ to the $2d$ (or $2d+1$) negative ones.

2. **The compact case.**

**Theorem 1.** Let $M$ be compact, $d$-dimensional, with metric $g$. There is an embedding $F: M \to \mathbb{R}^k_{2d}$, $k = d(d+5)/2$, which is isometric for $g$.

In [2], Nash shows that given a Riemannian metric $g$ on a compact manifold $M$, there is an isometric embedding of $M$ in euclidean $d(3d+11)/2$-dimensional space, where the euclidean space is flat Riemannian; the above theorem reduces the dimension required for the euclidean space and extends the result to arbitrary $g$, but the euclidean space is pseudo-Riemannian, even if $g$ is Riemannian.

3. **The noncompact case.** The following Theorem 2 derives the existence of isometric immersions for noncompact manifolds from the isometric immersions of compact ones; the lemma, applicable to compact or noncompact manifolds, then provides isometric embeddings in the noncompact case. More specific results are given in Theorem 3.

**Theorem 2.** Suppose that, for any metric $g$ on the $2d+1$-dimensional sphere $S^{2d+1}$, there is an isometric immersion of $S^{2d+1}$ in $\mathbb{R}^n$, where $\mathbb{R}^n$ has a flat Riemannian or pseudo-Riemannian metric. Then, if $M$ is any $d$-dimensional manifold with metric, there is an isometric immersion of $M$ in $\mathbb{R}^{2n}$, where $\mathbb{R}^{2n}$ has a flat Riemannian or pseudo-Riemannian metric. Furthermore, if, for any Riemannian metric $g$ on $S^{2d+1}$, there is an isometric immersion in $\mathbb{R}^{2n}$, then any manifold of dimension $d$ with a Riemannian metric has an isometric immersion in $\mathbb{R}^{2n}_{2d}$.

**Lemma.** If a $d$-dimensional manifold $M$, compact or not, has an isometric immersion in $\mathbb{R}^n$ for any metric $g$ on $M$, then $M$ has an isometric embedding in $\mathbb{R}^{n+2d}$ for any $g$. Further, if, for every Riemannian metric $g$, there is an isometric immersion in $\mathbb{R}^n$, then there is an isometric embedding in $\mathbb{R}^{n+2d+1}$.
Theorem 3. Every manifold $M$ of dimension $d$, compact or not, has some isometric embedding in $R^k$, $k = (2d + 1)(2d + 6)$, for every metric $g$ on $M$. If $g$ is a Riemannian metric, $M$ has an isometric embedding in $R^k$, $k = (2d + 1)(6d + 14)$.

Theorem 3 improves Nash's result that a noncompact $d$-dimensional manifold $M$ with Riemannian metric has an isometric embedding in $R^k$, $k = d(d + 1)(3d + 11)/2$.

4. The local case. If a metric $g$ is defined on an open set $U$ in a manifold $M$, then, given an open set $V$ with $V \subset U$, there is a metric $\tilde{g}$ defined on $M$ such that $g|_V = \tilde{g}|_V$. Thus the global statements above imply corresponding local conclusions. However, in the local case, using A. Friedman's results [1] on the local analytic case, we can considerably improve the dimensional requirements for the cases of Riemannian and pseudo-Riemannian metrics.

Theorem 4. Let $g$ be a metric defined on an open set in $R^d$ and $u$ a point of $U$. Suppose that $g$ has signature $(p_1, n_1)$ at $u$, $p_1 + n_1 = d$. Then there is an open set $V$, with $u \in V$, such that there is an embedding of $V$ in $\mathbb{R}^n$, $n + p = d(d + 3)/2$, which is isometric for $g|_V$. If $g$ is Riemannian at $u$, then $n$ may be taken equal to 0. More generally, $n$ and $p$ may be chosen subject only to the restrictions $n \geq n_1$, $p \geq p_1$, $n + p = d(d + 3)/2$.

References


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