STURM COMPARISON THEOREMS FOR ELLIPTIC INEQUALITIES

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Comparison theorems of Sturm's type will be stated for the quasilinear elliptic partial differential inequalities

\begin{align*}
(1) \quad & lu = - \sum_{i,j=1}^{n} D_i [a_{ij}(x, u) D_j u] + 2 \sum_{i=1}^{n} b_i(x, u) D_i u + uc(x, u) \leq 0, \\
(2) \quad & Lv = - \sum_{i,j=1}^{n} D_i [A_{ij}(x, v) D_j v] + 2 \sum_{i=1}^{n} B_i(x, v) D_i v + vC(x, v) \geq 0,
\end{align*}

where \( G \) is a nonempty regular bounded domain in \( \mathbb{R}^n \) and \( I \) is a real interval containing zero. The functions \( a_{ij}, A_{ij}, b_i, B_i, c, \) and \( C \) are assumed to be real-valued and continuous on \( G \times I \), and the matrices \( (a_{ij}) \) and \( (A_{ij}) \) symmetric and positive definite in \( G \times I \).

A Sturmian theorem has the following form: If (1) has a nontrivial solution \( u \) which vanishes identically on the boundary of \( G \) and if (2) majorizes (1) in some sense, then every solution \( v \) of (2) has a zero in \( G \) (or \( \overline{G} \)).

The linear selfadjoint case \( (b_i = B_i = 0, \ i = 1, \cdots, n) \) was first considered by Picone [12], and later independently and more generally by Hartman and Wintner [4], Kuks [10], Kreith [6], [8], Clark and Swanson [2]. A recent research announcement of Diaz and McLaughlin [3] is similar to Kreith's “strong comparison theorem” [9], obtained when \( \partial G \) has the “sphere property” by an appeal to the Hopf maximum principle. The conclusion of the strong comparison theorem is that \( v \) has a zero in \( G \) unless \( v \) is a constant multiple of \( u \); an analogous result in the quasilinear case is stated below (Theorem 2). Earlier McNabb [11] had used similar techniques in a different connection.

The linear nonselfadjoint case was studied by Protter [13], Swanson [16], Kreith [9], and Allegretto [1]. Extensions to unbounded domains were obtained in [16] and [17] and applied to oscillation theory and eigenvalue estimation. Comparison theorems

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1318
in terms of eigenvalues associated with boundary problems for differential operators also have been developed [7], [8], [15] and used to derive oscillation criteria [1], [5]. The quasilinear case was considered by Redheffer [14] and the authors [1], [19]. An extensive bibliography on comparison and oscillation theory can be found in [18].

Let $m$, $M$ denote the $(n+1)$-square matrix functions on $\mathbb{C} \times I$ defined by

$$m(x, u) = \begin{pmatrix} (a_{ij}(x, u)) & (b_i(x, u))^T \\ (b_i(x, u)) & c(x, u) \end{pmatrix},$$

$$M(x, u) = \begin{pmatrix} (A_{ij}(x, u)) & (B_i(x, u))^T \\ (B_i(x, u)) & C(x, u) + H(x, u) \end{pmatrix}$$

respectively, where

$$H(x, u) = - \left[ \det(A_{ij}(x, u)) \right]^{-1} \sum_{i=1}^n B_i(x, u) B_i^*(x, u),$$

$B_i^*(x, u)$ denoting the cofactor of $B_i(x, u)$ in the matrix $M(x, u)$.

By means of Green's formula the following functional is associated with $l$ in a natural way

$$f[u] = \int_G \left[ \sum_{i,j} a_{ij}(x, u(x)) D_i u D_j u + 2u \sum_i b_i(x, u(x)) D_i u \\
+ u^2 c(x, u(x)) \right] dx.$$

The domain $\mathcal{D}$ of $f$ is defined to be the set of all real-valued functions $u \in C^1(\mathbb{C})$ with range in $I$ such that $u$ vanishes identically on $\partial G$.

**Theorem 1.** If

1. there exists a function $u \in \mathcal{D}$ such that $f[u] \leq 0$ (respectively, $f[u] < 0$);
2. $Lu \geq 0$ throughout $G$;
3. $v(x) > 0$ for some $x \in G$;
4. $m(x, u(x)) - M(x, v(x))$ is positive definite (respectively, positive semidefinite) for all $x \in G$;

then $v$ must vanish at some point in $\mathcal{G}$. The same conclusion holds if the inequalities in (2) and (3) are replaced by $Lv \leq 0$ and $v(x) < 0$. The same conclusion holds if (2) and (3) are replaced by $Lv = 0$ throughout $G$.

It follows from Green's formula that hypothesis (1) is implied by the existence of a solution $u$ of $lu \leq 0$ (respectively, $lu \geq 0$) satisfying $u > 0$ (respectively, $u < 0$) throughout $G$ and $u = 0$ on $\partial G$.
The "strong" conclusion that $v$ must in fact vanish at some point in $G$ can be obtained by our methods [1], [19] under additional assumptions. Also, an analogue of Theorem 1 is available when the coefficients $a_{ij}, b_i, c$, etc. are functions of first or higher order derivatives of $u$. These results with proofs will appear elsewhere.

Of special interest in oscillation theory are cases for which the hypotheses of Theorem 1 are satisfied when $l$ is linear. In such cases, the known properties of linear symmetric operators can be employed to describe the oscillatory behaviour of $L$. Simple examples which can be treated this way are Mathieu's and Duffing's equations.

The pointwise inequality in hypothesis (4) of Theorem 1 can be replaced by a weaker integral inequality of the type given in [2], [15], [16], and [17]. For simplicity, we shall state our result in the selfadjoint case $b_i = B_i = 0$ identically, $i = 1, \ldots, n$. Let $F[u]$ denote the analogue of the functional (3) for $L$, i.e. with $a_{ij}$ and $c$ in (3) replaced by $A_{ij}$ and $C$, respectively.

THEOREM 2 (SELFADJOINT CASE). Suppose that $L$ is uniformly elliptic in a nonempty regular bounded domain $G$ whose boundary has bounded curvature, and that the matrix function $v \rightarrow M(x, v)$ is nonincreasing (as a form) on $I$ for each $x \in G$. If there exists a nontrivial solution $u \in \mathcal{D}$ of (1) such that $u > 0$ in $G$ and $f[u] \geq F[u]$, then every solution $v$ of (2) has one of the following properties:

(i) There exists a subdomain $G_v \subset G$ such that $v(x) < u(x)$ for all $x \in G_v$, or

(ii) $v$ is a constant multiple of $u$.

An example given in [19] shows that the conclusion of Theorem 2 is false without the nonincreasing hypothesis on $M$. In the linear case, conclusion (i) is strengthened to

(i') $v(x)$ vanishes at some point $x \in G$ (Kreith's theorem [8]).

Theorem 2 can be used to obtain nonoscillation criteria of the Kneser-Hille-Glazman type [19].

REFERENCES


