NORMS ON QUOTIENT SPACES
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Communicated by Bertram Yood, June 23, 1967

1. Perturbation classes. Let $S$ be a subset of a Banach space $\mathfrak{a}$ over the complex numbers, and assume that $aS \subseteq S$ for each scalar $a \neq 0$. Let $P(S)$ denote the set of elements of $\mathfrak{a}$ that perturb $S$ into itself, i.e., $P(S) = \{ a \in \mathfrak{a} : a + s \in S \text{ for all } s \in S \}$.

**Proposition 1.1.** $P(S)$ is a linear subspace of $\mathfrak{a}$. If $S$ is an open subset of $\mathfrak{a}$, then $P(S)$ is closed.

**Proposition 1.2.** Let $S_1 \subseteq S_2$ be two such subsets, and assume that $S_1$ is open and $S_2$ does not contain any boundary point of $S_1$. Then $P(S_2) \subseteq P(S_1)$.

**Proposition 1.3.** Assume that $\mathfrak{a}$ is a Banach algebra with identity $e$. Let $G$ denote the set of invertible elements in $\mathfrak{a}$. If $G \subseteq S$, then $P(S)$ is a left ideal. If $G \subseteq S$, then $P(S)$ is a right ideal.

**Proposition 1.4.** $P(G) = P$, the radical of $\mathfrak{a}$.

Let $G_l$ ($G_r$) denote the set of left (right) invertible elements of $\mathfrak{a}$, and let $H_l$ ($H_r$) denote the set of elements of $\mathfrak{a}$ that are not left (right) topological divisors of zero.

**Theorem 1.5.** $P(H_l) \subseteq P(G_l) = R = P(G_r) \supseteq P(H_r)$.

Let $X$ be a Banach space, and let $B(X) [K(X)]$ denote the set of bounded (compact) linear operators on $X$. Take $\mathfrak{a} = B(X)/K(X)$ and let $\pi$ be the canonical homomorphism from $B(X)$ to $\mathfrak{a}$. Set

$$\Phi(X) = \pi^{-1}(G_l), \quad \Phi_l(X) = \pi^{-1}(G_l), \quad \Phi_r(X) = \pi^{-1}(G_r).$$

It is well known [6] that $\Phi_l(X)$ consists of those operators having finite nullity and closed, complemented ranges, and that $\Phi_r(X)$ consists of those operators having complemented null spaces and closed ranges with finite codimensions. $\Phi(X) = \Phi_l(X) \cap \Phi_r(X)$ is the set of Fredholm operators on $X$.

**Theorem 1.6.** $P(\Phi) = P(\Phi_l) = P(\Phi_r) = \pi^{-1}(R)$.

Let $Z$ be any subset of $\{ 0, \pm 1, \pm 2, \ldots, \pm \infty \}$, and let $\Phi_z$ be the collection of those operators $A \in \Phi_l(X) \cup \Phi_r(X)$ such that $i(A) \in Z$, where $i(A) = \dim N(A) - \dim N(A')$.

**Theorem 1.7.** $P(\Phi_z) = \pi^{-1}(R)$.
2. Measures of noncompactness. Let $X$, $Y$ be Banach spaces, and denote the set of bounded (compact) linear operators from $X$ to $Y$ by $B(X, Y)$ [$\mathcal{K}(X, Y)$]. Let $S_X$ denote the unit ball in $X$. For any bounded subset $\Omega$ of $X$ let $q(\Omega)$ denote the greatest lower bound of the numbers $r$ such that $\Omega$ can be covered by a finite collection of spheres of radius $r$. For $A \in B(X, Y)$ set $\|A\|_q = q[A(S_X)]$. Let $\|A\|_m$ denote the greatest lower bound of all numbers $\eta$ such that $\|Ax\| \leq \eta \|x\|$ for all $x$ in some subspace having finite codimension. Let $\pi$ denote the canonical homomorphism of $B(X, Y)$ into $B(X, Y)/\mathcal{K}(X, Y)$.

**Proposition 2.1.** Both $\|\cdot\|_q$ and $\|\cdot\|_m$ are seminorms and satisfy
$$\|BA\|_q \leq \|B\|_q \|A\|_q, \quad \|BA\|_m \leq \|B\|_m \|A\|_m, \quad \|A\|_q \leq \|\pi(A)\|, \quad \|A\|_m \leq \|\pi(A)\|, \quad \|A + K\|_q = \|A\|_q, \quad \|A + K\|_m = \|A\|_m$$
for $K \in \mathcal{K}(X, Y)$.

**Theorem 2.2.** $\|A\|_q/2 \leq \|A\|_m \leq 2\|A\|_q$.

**Definition 2.3.** A Banach space $X$ will be said to have the compact approximation property with constant $\gamma$ if for each $\epsilon > 0$ and finite set of points $x_1, \ldots, x_n$ in $X$ there is an operator $K \in \mathcal{K}(X)$ such that $\|I - K\| \leq \gamma$ and $\|x_j - Kx_j\| < \epsilon$ for $1 \leq j \leq n$.

**Theorem 2.4.** If $Y$ has the compact approximation property with constant $\gamma$, then $\|\pi(A)\| \leq \gamma \|A\|_q$. Thus $B(X, Y)/\mathcal{K}(X, Y)$ is complete with respect to the norms induced by $\|\cdot\|_q$ and $\|\cdot\|_m$.

3. Semi-Fredholm operators. An operator $A \in B(X, Y)$ is in $\Phi_+(X, Y)$ if it has finite nullity and closed range.

**Theorem 3.1.** An operator $A$ is in $\Phi_+(X, Y)$ if and only if for each Banach space $Z$ there is a constant $C$ such that $\|T\|_m \leq C\|AT\|_m$, $T \in B(Z, X)$. The constant does not depend on $Z$.

**Corollary 3.2.** If $A \in \Phi_+(X, Y)$ and $X$ has the compact approximation property, then $\|\pi(T)\| \leq C\|\pi(AT)\|$, $T \in B(Z, X)$, for any Banach space $Z$.

**Definition 3.3.** For $A \in B(X, Y)$ set $q_A = \text{glb} \ q[A(\Omega)]/q(\Omega)$, where the glb is taken over all bounded subsets $\Omega$ of $X$.

**Theorem 3.4.** $A \in \Phi_+(X, Y)$ if and only if $q_A \neq 0$.

An operator $A \in B(X, Y)$ is in $\Phi(X, Y)$ if its range is closed and has finite codimension.

**Theorem 3.5.** $A \in \Phi_-(X, Y)$ if and only if $\beta(A - K) < \infty$ for all $K \in \mathcal{K}(X, Y)$, where $\beta(E) = \text{codim} \ R(E)$.
Theorem 3.6. \( A \in \Phi_-(X, Y) \) if and only if for each \( Z \) there is a constant \( C \) such that \( \| T \|_m \leq C \| TA \|_m \), \( T \in B(Y, Z) \). The constant \( C \) is independent of \( Z \).

We now consider the case \( X = Y \). Let \( r_\sigma(A) \) denote the spectral radius of an operator \( A \).

Theorem 3.7. If \( \| A^n \|_m < 1 \) for some \( n \geq 1 \), then \( I - A \in \Phi(X) \) and \( i(I - A) = 0 \).

Theorem 3.8.
\[
r_\sigma[\pi(A)] = \lim_{n \to \infty} \| A^n \|_m^{1/n} = \lim_{n \to \infty} \| A^n \|_q^{1/n} = \max_{\lambda \in \sigma_e(A)} | \lambda |,
\]
where \( \sigma_e(A) \) denotes the essential spectrum of \( A \) according to any of the usual definitions [8], [9].

Corollary 3.9. \( r_\sigma[\pi(A)] \geq q_A \). Hence an operator in \( \Phi_+(X) \) cannot be a Riesz operator.

Definition 3.10. A space \( X \) has the range property if for each \( \epsilon > 0 \) and each \( A \in B(X) \) with \( \dim N(A) = \infty \) there is a \( T \in B(X) \) such that \( \| T \|_q = 1 \) and \( q[T(S_x) \setminus N(A)] < \epsilon \). All subprojective [10] spaces have the range property.

Theorem 3.11. If \( X \) has the range property, then \( A \in \Phi_+(X) \) if and only if \( \| T \|_q \leq C \| AT \|_q \) for all \( T \in B(X) \).

Theorem 3.12. If \( X \) is subprojective and \( \pi(A) \) is not a left zero divisor then \( A \in \Phi_+(X) \).

Corollary 3.13. If \( X \) is subprojective and has the compact approximation property, then every topological left zero divisor in \( B(X)/\mathcal{K}(X) \) is a left zero divisor.

Theorem 3.14. If \( X \) is superprojective [10] and \( \pi(A) \) is not a right zero divisor, then \( A \in \Phi(X) \).

Corollary 3.15. If \( X \) is both subprojective and superprojective, then every element of \( B(X)/\mathcal{K}(X) \) which is not a zero divisor is invertible.

4. Remarks. Some of the results of §1 were also obtained by B. Gramsch [12]. The \( q \)-seminorm was studied by Gol'denšteĩn, Gokhberg, Markus [1], [2] and Darbo [3]. The basic idea goes back to Kuratowski [11]. For the \( q \)-seminorm Proposition 2.1 was proved in [1]. The compact approximation property is weaker than the metric approximation property of Grothendieck [4] and is similar to one of Bonsall [5].
BIBLIOGRAPHY


5. Frank F. Bonsall, Compact linear operators, Lecture notes, Yale University, New Haven, Conn., 1967.


