FACTORIZATION OF OPERATOR VALUED ENTIRE FUNCTIONS

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Let \( W(z) \) be a complex valued entire function of exponential type with nonnegative values on the real axis. We call \( W(z) \) factorable if

\[
W(z) = A^*(z)A(z)
\]

where \( A(z) \) is an entire function whose restriction to the upper half-plane is an outer function. Here \( A^*(z) = A(z) \). Recall that an outer function in the upper half-plane is a function of the form

\[
f(z) = C \exp \left( \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{1 + tz \log k(t)}{t - z} \frac{dt}{1 + t^2} \right), \quad y > 0,
\]

where \( C \) is a constant of absolute value 1, \( k(t) \geq 0 \) a.e. on \( (-\infty, \infty) \), and \( (1+t^2)^{-1} \log k(t) \in L^1(-\infty, \infty) \). Of necessity, \( k(x) = \lim_{y \to 0} \left| f(x+iy) \right| \) a.e. where the limit is taken as \( y \) decreases to zero. Therefore the restriction of an entire function \( A(z) \) to the upper half-plane is an outer function if and only if \( (1+t^2)^{-1} \log \left| A(t) \right| \in L^1(-\infty, \infty) \) and

\[
\log \left| A(z) \right| = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\log \left| A(t) \right|}{(t-x)^2 + y^2} \, dt, \quad y > 0.
\]

The following facts are available from the classical theory of entire functions:

(1°) for \( W(z) \) to be factorable, it is necessary and sufficient that

\[
\int_{-\infty}^{\infty} \frac{\log + W(x)}{1 + x^2} \, dx < \infty,
\]

(2°) if \( W(z) \) is factorable, the factor \( A(z) \) is determined to within a multiplicative constant of absolute value 1,

(3°) if \( W(z) \) is factorable, say \( W(z) = A^*(z)A(z) \) as above, and if \( W(z) \) is of exponential type \( \tau \), then \( \exp(-\frac{1}{2}i\pi z)A(z) \) is of exponential type \( \frac{1}{2} \tau \). See [2, p. 125], [3, p. 34], and [4, p. 437], where some original sources are cited. The purpose of this note is to communicate extensions of these results to operator valued entire functions.

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Let $\mathcal{C}$ be a separable complex Hilbert space. By a vector or operator valued function we shall mean a function whose values are vectors in $\mathcal{C}$ or bounded operators on $\mathcal{C}$ respectively. Analyticity is defined in the weak sense. The bar of a bounded operator on $\mathcal{C}$ denotes its adjoint, and we use the notation $A^*(z) = \overline{A(\overline{z})}$ for operator as well as scalar valued entire functions. By $H^2_e$ we mean the Hardy space of vector valued analytic functions $f(z)$ defined for $y > 0$ such that

$$||f||^2 = \lim_{y \to 0} \int_{-\infty}^{+\infty} ||f(x + iy)||^2 dx < \infty.$$ 

If $\mathfrak{B}$ is a closed subspace of $\mathcal{C}$, $H^2_{\mathfrak{B}}$ denotes the closed subspace of $H^2_e$ of functions with values in $\mathfrak{B}$. An operator valued analytic function $A(z)$ defined for $y > 0$ is called outer if there exists a bounded scalar valued outer function $f(z)$ such that

(i) $B(z) = f(z)A(z)$ is bounded for $y > 0$, and

(ii) the range of multiplication by $B(z)$ in $H^2_e$ is dense in a subspace of the form $H^2_{\mathfrak{B}}$.

In this case the closed subspace $\mathfrak{B}$ of $\mathcal{C}$ implied in (ii) does not depend on the choice of $f(z)$, and we say that $A(z)$ acts in $\mathfrak{B}$. An operator valued entire function $f(z)$ is said to be of exponential type $\tau$, $\tau \geq 0$, if for every vector $c$ in $\mathcal{C}$ the scalar valued entire function $f_c(z) = \langle f(z), c \rangle_e$ is of exponential type $\tau$. An operator valued entire function $W(z)$ is said to be of exponential type if for every vector $c$ in $\mathcal{C}$ the vector valued entire function $W(z)c$ is of exponential type $\tau(c)$ for some number $\tau(c) \geq 0$.

Let $W(z)$ be an operator valued entire function of exponential type which has nonnegative values on the real axis. We call $W(z)$ factorable if

$$W(z) = A^*(z)A(z)$$

where $A(z)$ is an operator valued entire function whose restriction to the upper half-plane is an outer function. We can now state our main results. Let $I$ denote the identity operator on $\mathcal{C}$.

**Theorem 1.** Let $W(z)$ be an operator valued entire function of exponential type which has nonnegative values on the real axis. Assume that there exists a scalar valued entire function $w(z)$ of exponential type such that

$$W(z) \leq w(z)I$$

for all real $x$. If $w(z)$ is factorable, then so is $W(z)$. 

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Theorem 2. Let \( W(z) \) be an operator valued entire function of exponential type which has nonnegative values on the real axis. For each \( j = 1, 2 \), let
\[
W(z) = A_j^*(z) A_j(z)
\]
where \( A_j(z) \) is an operator valued entire function whose restriction to the upper half-plane is an outer function, and let this outer function act in the subspace \( \mathcal{B}_j \) of \( \mathbb{C} \). Then
\[
A_j(z) = UA_1(z)
\]
where \( U \) is a partially isometric operator on \( \mathbb{C} \) with initial set \( \mathcal{B}_1 \) and final set \( \mathcal{B}_2 \).

Theorem 3. Let \( W(z) \) be an operator valued entire function of exponential type which has nonnegative values on the real axis. Assume that \( W(z) \) admits a scalar dominant as in Theorem 1. Assume that the scalar dominant is factorable, and let
\[
W(z) = A^*(z) A(z)
\]
where \( A(z) \) is an operator valued entire function whose restriction to the upper half-plane is an outer function. If for some vector \( c \) in \( \mathbb{C} \), \( W(z)c \) is of exponential type \( \tau(c) \), \( \tau(c) \geq 0 \), then \( \exp(-\frac{1}{2}i\tau(c)z)A(z)c \) is of exponential type \( \frac{1}{2}\tau(c) \).

Consider the case \( \dim \mathbb{C} < \infty \). Let \( W(z) \) be an entire function of exponential type which has nonnegative values on the real axis. If
\[
\int_{-\infty}^{+\infty} \log^+[\text{tr} W(x)] \frac{dx}{1 + x^2} < \infty,
\]
then \( W(z) \) is factorable. For in Theorem 1 we may choose \( w(z) = \text{tr} W(z) \).

Another corollary is valid for an arbitrary separable coefficient space \( \mathbb{C} \). Let \( W(z) = C_0 + C_1 z + \cdots + C_{2N} z^{2N} \) be a polynomial with operator coefficients which has nonnegative values on the real axis. Then
\[
W(z) = A^*(z) A(z)
\]
where \( A(z) = A_0 + A_1 z + \cdots + A_N z^N \) is a polynomial with operator coefficients whose restriction to the upper half-plane is an outer function.

Proofs of these results will appear elsewhere. Theorem 2 is a special
case of a more general assertion concerning outer functions. It is
deduced from [6, Theorem 3]. In the case where $W(z)$ is bounded on
the real axis, i.e. when we can choose $w(z)$ to be a positive constant,
Theorems 1 and 3 are proved using a Hilbert space method originated
by D. Lowdenslager [5] and developed by the first author [6]. The
general cases of Theorems 1 and 3 are obtained from this special
case by means of a theorem of A. Beurling and P. Malliavin [1].

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