QUASI-SUBORDINATION AND COEFFICIENT CONJECTURES[1]

BY M. S. ROBERTSON

Abstract. The concepts of subordination and majorization of two analytic functions are unified by regarding them as special cases of the idea of quasi-subordination. Six conjectures are discussed in connection with quasi-subordination, univalent and multivalent functions. Evidence is given to support the truth of these conjectures.

1. **Subordination.** Let D be a simply connected domain on the z-sphere and let $w=F(z)$ be meromorphic on D and map D onto $D(F)$ the Riemann domain over the w-sphere.

Let $w=f(z)$ be also meromorphic in D.

DEFINITION. $f(z)$ is called *subordinate* to $F(z)$ in D, with center z_0 in D, if $f(z_0)=F(z_0)$ and the values of $f(z)$ in D, determined by analytical continuation from z_0, are situated in $D(F)$.

The Riemann domain $D(f)$ is extended (but is not necessarily schlicht) over $D(F)$.

We write $f(z)\prec F(z)$ in D.

There is no loss in generality in assuming D to be the unit disc $E\{z|\ |z|<1\}$ and $z_0=0$. Under these assumptions there exists a function $w(z)$ regular in $|z|<1$ with $|w(z)|\leqq|z|<1$ such that $f(z)=F\{w(z)\}\,(|z|<1)$.

In our present discussion we shall be concerned with the case that $f(z)$ and $F(z)$ are both regular in F. Frequently we shall take $f(0)=F(0)=0$.

2. **Majorization.** If $f(z)$ and $F(z)$ are both regular in E, and if

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$|f(z)| \leqq |F(z)|$ in E, then there exists a bounded function $\phi(z)$ regular in E, $|\phi(z)| \leqq 1$ in E, for which

$$f(z) \equiv \phi(z)\cdot F(z) \qquad (z \in E).$$

In this case we say $f(z)$ is majorized by $F(z)$ and we write $f(z) \ll F(z)$.

The concepts of subordination and majorization often play dual or related roles in many theorems involving analytic functions. We mention a few examples.

EXAMPLE 1. Let $f(z)=\sum_1^\infty a_n z^n \prec z+\sum_2^\infty A_n z^n = F(z)$, $F'(0)=1$, $z\in E$, where $F(z)$ is typically-real in E. This implies that $F(z)$ is real on the real axis, the coefficients A_n are real, and, for $z\in E$, $F(z)$ takes on real values only for real values of z. $I_m F(z)>0$ whenever $I_m z>0$, $z\in E$.

W. Rogosinski [29] showed that when $F(z)$ is typically-real in E then $|a_n| \leqq n$, $n=1, 2, \cdots$ for the coefficients a_n of $f(z)$. Recently T. H. MacGregor [18] proved the same result with subordination of f by F replaced by majorization $|f(z)| \leqq |F(z)|$ in E.

EXAMPLE 2. Let $f(z)\prec F(z)$, $f(0)=F(0)=0$, $F(z)$ univalent in E, i.e., $F(z_2)\neq F(z_1)$ for $z_2\neq z_1$, z_1 and z_2 in E. Let $\arg f'(0)=\arg F'(0)$. M. Biernacki [2] has shown that if $f(z)\not\equiv F(z)$ there is a number $R_0 \geqq 1/4$ so that for $|z|<R_0 |f(z)|<|F(z)|$. The estimate for R_0 was improved by G. M. Golusin [8] who obtained $0.35<R_0\leqq(3-\sqrt{5})/2=.381\cdots$. Finally Shah Tao-shing [30] showed that $R_0=(3-\sqrt{5})/2$. G. M. Golusin [7] also showed that if $f(z)$ is univalent in E then $R_0=0.39\cdots$ where

$$\log((1+R_0)/(1-R_0)) + 2 \arctan R_0 = \pi/2.$$

Z. Lewandowski [13] has proved a converse theorem. If f and F are regular in E, $f(0)=F(0)=0$, F univalent in E, and if $|f|\leqq|F|$ in E then for $|z|<R_1$ $f\prec F$ where $.21<R_1<.30$. If $\arg f'(0)=\arg F'(0)>0$ and F is starlike then $R_1=R^*$ where $R^{*3}+R^{*2}+3R^*-1=0$ and $.29<R^*<.30$.

EXAMPLE 3. Let $\{t_n\}$, $n=1, 2, \cdots$ be a sequence of positive numbers with $\lim_{n\to\infty} t_n=t_0=0$. Let $F(z, t_n)$ be a sequence of functions $(n=0, 1, \cdots)$ regular in E, $F(0, t_n)=0$, $F(z, t_0)=F(z, 0)=f(z)$. Let $F(z, t_n)\prec f(z)$ in E. Let

$$F(z) = \lim_{n\to\infty} \frac{F(z, t_n)-f(z)}{t_n}$$

exist $(z\in E)$. Then $\operatorname{Re}\{F(z)/zf'(z)\}\leqq 0$ $(z\in E, f'(z)\neq 0)$. The author [24] proved this theorem with the hypothesis that $f(z)$ is univalent in E since the interesting applications are in this case. However, the theorem extends easily to all $f(z) \neq$ constant function, $f(z)$ regular in E.

To illustrate one simple application, let $0<t_n \leqq 1$, $f(z)$ regular and nonconstant in E. Let $(1-t_n)\cdot f(z)=F(z, t_n) \prec f(z)$ in E; then

$$F(z) = \lim_{n\to\infty}\left[\frac{(1-t_n)\cdot f(z) - f(z)}{t_n}\right] = -f(z)$$

and we conclude that $\operatorname{Re}\{f(z)/zf'(z)\} \geqq 0$. Equality can only occur when $f(z)/zf'(z)=ai$, a real. But $f(z)/zf'(z)=1/q+\cdots$ for $f(z) = a_q z^q + \cdots$, $q \geqq 1$. Hence $a=0$. Then $f(z)=$constant. We conclude that $\operatorname{Re}\{zf'(z)/f(z)\}>0$ in E. If also $f'(0)\neq 0$ we have the familiar test that $f(z)$ be univalent and starlike in E.

To illustrate another application of this theorem, suppose that $f(z)=\sum_1^\infty a_n z^n$ is regular and univalent in E and that the de la Vallée Poussin polynomials $V_n(z)$ approximating $f(z)$ are subordinate to $f(z)$, $V_n(z) \prec f(z)$ in E, $n=1, 2, \cdots$

$$V_n(z) = \sum_{k=1}^{n} \frac{n(n-1)\cdots(n-k+1)}{(n+1)(n+2)\cdots(n+k)}\cdot a_k z^k.$$

We take $t_n=1/(n+1)$, $F(z, t_n)=V_n(z)$ and compute $F(z)$ to be $-[z^2f''(z)+zf'(z)]$ so that $\operatorname{Re}\{F(z)/zf'(z)\} \leqq 0$ leads to

$$\operatorname{Re}\{1 + zf''(z)/f'(z)\} \geqq 0 \qquad (z \in E).$$

We conclude that $f(z)$ must be convex in E. Previously Pólya and Schoenberg [20] had proved the converse to this theorem. Thus the necessary and sufficient condition that a function $f(z)$, regular and univalent in E, should map E on a convex domain is that $V_n(z) \prec f(z)$, $n=1, 2, \cdots$.

Z. Lewandowski [14] has also extended this theorem replacing subordination of $F(z, t_n)$ by $f(z)$ by majorization $|F(z, t_n)| \leqq |f(z)|$ and obtains a conclusion $\operatorname{Re}\{F(z)/f(z)\} \leqq 0$.

The three examples I have presented illustrate the interrelation of the concepts of subordination and majorization. In order to attempt a unification of the two concepts I introduce the concept of quasi-subordination.

Definition. Let $f(z)=\sum_0^\infty a_n z^n$, $F(z)=\sum_0^\infty A_n z^n$ be analytic in $|z|<R$. Let $\phi(z)$ be a function analytic and bounded for $|z|<R$, $|\phi(z)| \leqq 1$, such that $f(z)/\phi(z)$ is regular and subordinate to $F(z)$ in $|z|<R$. Then $f(z)$ is said to be quasi-subordinate to $F(z)$ in $|z|<R$ relative to $\phi(z)$. We write $f(z) \prec_q F(z)$ and $f(z)=\phi(z)F(w(z))$, $|w(z)| \leqq |z| < R$.

Two special cases of quasi-subordination are of particular interest.

(1) If $\phi(z)\equiv 1$ then $f(z)\prec F(z)$, $|z|<R$.

(2) If $w(z)\equiv z$, then $f\ll F(z)$ for $|z|<R$. In what follows, we shall assume $R=1$. Note that if $f\prec_q F(z)$ then $a_0=\phi(0)\cdot A_0$ since $f(0)/\phi(0)=F(0)$.

I turn now to the problem of relating the relative magnitudes of the coefficients a_n and A_n when $f\prec_q F$.

If $f(z)=\sum_0^\infty a_n z^n \prec F(z)=\sum_0^\infty A_n z^n$ (ordinary subordination). J. E. Littlewood [15] showed that $|a_1|\leqq|A_1|$ and $|a_2|\leqq \operatorname{Max}\{|A_1|, |A_2|\}$ and the bounds are sharp. On the other hand, if $f\prec_q F$ then $|a_1|\leqq|A_1|$ again, but $|a_2|\leqq(5/4)\operatorname{Max}\{|A_1|, |A_2|\}$. The constant 5/4 is best possible [27] with equality when $A_1=A_2$ and $f(z)=((1+2z)/(2+z))\cdot F(z)$. Here $|f|\leqq|F|$ so $f\prec_q F$ in E. If $f\prec F$, $\rho>0$,

$$\int_0^{2\pi}|f(re^{i\theta})|^\rho d\theta \leqq \int_0^{2\pi}|F(re^{i\theta})|^\rho d\theta, \qquad 0\leqq r<1,$$

and the same is true when $f\prec_q F$. W. Rogosinski [29] showed that if $f\prec F$ then

$$\sum_{k=0}^{n}|a_k|^2 \leqq \sum_{k=0}^{n}|A_k|^2, \qquad n=1, 2, \cdots [a_0=A_0].$$

The same inequalities hold [27] if $f\prec_q F$ in E. Also if $F(0)\neq 0$, and in the neighborhood of $z=0$ the functions $[f(z)/\phi(z)]^{1/2}$, $[F(z)]^{1/2}$ (which may not be regular in E) have the expansions

$$\left[\frac{f}{\phi}\right]^{1/2} = \sum_0^\infty b_k z^k, \qquad [F]^{1/2} = \sum_0^\infty B_k z^k \quad (|z| \text{ small})$$

then

$$\sum_0^n |b_k|^2 \leqq \sum_0^n |B_k|^2, \qquad n=0, 1, \cdots.$$

This basic inequality permits us to obtain bounds on a_n when $f=\sum_1^\infty a_n z^n \prec_q F=\sum_0^\infty A_n z^n$, $A_0\neq 0$. We get $|a_n|\leqq\sum_{k=0}^{n-1}|B_k|^2$, $n=1, 2, \cdots$ where $[F(z)]^{1/2}=\sum_0^\infty B_k z^k$, $B_0\neq 0$, near $z=0$.

The proof is not difficult:

$$f(z)/\phi(z) = F\{w(z)\}, \quad |w(z)| \leqq |z| < 1.$$

Let $[f/\phi]^{1/2}=\sum_0^\infty b_k z^k$, $b_0=B_0\neq 0$, near $z=0$.

$$f(z) = \phi(z)\cdot\left(\sum_0^\infty b_n z^n\right)^2.$$

For small r,

$$a_n = \frac{1}{2\pi i}\int_{|z|=r} \frac{f(z)}{z^{n+1}}\,dz = \frac{1}{2\pi i}\int_{|z|=r} \phi(z)\left(\sum_0^\infty b_n z^n\right)^2 \frac{dz}{z^{n+1}}$$

$$= \frac{1}{2\pi i}\int_{|z|=r} \phi(z)\left(\sum_{k=0}^{n-1} b_k z^k\right)^2 \frac{dz}{z^{n+1}}, \qquad \text{for all } r < 1,$$

where we have used the fact that $\phi(0)=0$. For $z=re^{i\theta}$, $0\leqq r<1$,

$$|a_n| \leqq \frac{1}{2\pi r^{n-1}}\int_0^{2\pi} \left|\sum_{k=0}^{n-1} b_k z^k\right|^2 d\theta = \frac{1}{r^{n-1}}\sum_{k=0}^{n-1} |b_k|^2 r^{2k}, \qquad 0\leqq r<1,$$

or

$$|a_n| \leqq \sum_0^{n-1} |b_k|^2 \leqq \sum_0^{n-1} |B_k|^2.$$

In particular, if $g(z)$ is univalent in E and $g(z)=\sum_1^\infty c_n z^n$, $c_1\neq 0$, we may take $F(z)$ above to be $g(z)/z$ and if $f(z)\prec_q g(z)$ in E, then $f\prec_q F=g/z$. Then $[g(z^2)]^{1/2}=\sum_{k=1}^\infty d_{2k-1}z^{2k-1}$ is univalent in E, and $[F(z)]^{1/2}=[g(z)/z]^{1/2}=\sum_1^\infty d_{2k-1}z^{k-1}$. Thus $|a_n|\leqq\sum_0^{n-1}|B_k|^2$ becomes $|a_n|\leqq\sum_{k=1}^n |d_{2k-1}|^2$. Thus the problem of bounding $|a_n|$ when $f(z)=\sum_1^\infty a_n z^n \prec_q g(z)=\sum_1^\infty c_n z^n$ in E, and when $g(z)$ is univalent in E, becomes one of determining the sharp upper bound on $\sum_{k=1}^n |d_{2k-1}|^2$ for the associated univalent function $[g(z^2)]^{1/2}=\sum_0^\infty d_{2k+1}z^{2k+1}$.

Let us return to the subordination situation $f(z)=\sum_1^\infty a_n z^n \prec F(z)=\sum_1^\infty A_n z^n$ in E, with $F(z)$ univalent in E, i.e., $F(z_2)\neq F(z_1)$ for $z_2\neq z_1$ in E. There are three long-standing and still unproved conjectures regarding the coefficients of A_n of the power series for $F(z)$.

CONJECTURE I (BIEBERBACH, 1916). If $F(z)=\sum_1^\infty A_n z^n$ in E, then $|A_n|\leqq n|A_1|$, $n=2, 3, \cdots$. In 1916 the conjecture was proved to be true for $n=2$ by Bieberbach [**1**]; in 1923 for $n=3$ by Löwner [**17**], and then after a very long gap, in 1955 for $n=4$ by Garabedian and Schiffer [**6**]. Recently Pederson [**19**] has succeeded in proving the conjecture for $n=6$. It is not known to be true for $n=5$ and $n>6$, except that for each $F(z)$ there is an $N_0(F)$ so that $|A_n|\leqq n|A_1|$ for $n>N_0(F)$, an important result due to W. Hayman [**11**].

The conjecture is known to be true for all n if $F(z)$ satisfies any one of the following additional conditions:

(1) $F(z)=\sum_1^\infty A_n z^n$ has A_n all real (Dieudonné [**4**] and Rogosinski [**28**].

(2) $F(z)$ maps E on a spirallike domain (Špaček [**31**]).

(3) $F(Z)$ maps E on a domain D starlike with respect to a point w_0 exterior to D (N. G. de Bruijn [**3**]).

(4) $F(z)$ is close-to-convex in E (M. O. Reade [21]).

CONJECTURE II (ROBERTSON, 1936). If $F=\sum_1^\infty A_n z^n$ is regular and univalent in E, and if $[F(z^2)]^{1/2}=\sum_1^\infty D_{2k-1}z^{2k-1}$ then

$$(1) \qquad \sum_{k=1}^{n} |D_{2k-1}|^2 \leqq n|A_1|, \qquad n=2,3,\cdots, \quad A_1=D_1^2.$$

Since $|D_3|\leqq|D_1|$ follows from $|A_2|\leqq 2|A_1|$, the inequality (1) is certainly true for $n=2$ (and trivially for $n=1$). However, since $|D_5|$ can be as large as $(e^{-2/3}+\frac{1}{2})\cdot|D_1|>|D_1|$ it is somewhat surprising to find that (1) is still true for $n=3$. The author [22] proved this in 1936, using the Löwner variational method of attack. Although three decades have passed it is still not known if (1) is true for larger values of n, except that only recently S. Friedland [5] succeeded in establishing (1) in the case $n=4$. The truth of Conjecture II would imply the truth of the Bieberbach Conjecture I. There is additional evidence that (1) may be true. For example, if $F(z)$ is starlike or spirallike then $|D_{2k-1}|\leqq|A_1|^{1/2}$ for all k so that (1) is true for all n in this case. Moreover, $\lim_{n\to\infty}|D_{2k-1}|=\alpha$ exists and $\alpha<|D_1|$ except when $F(z)=z(1-\epsilon z)^{-2}$, $|\epsilon|=1$, a result due to W. Hayman [11]. One can show from this that the conjecture (1) is true for $n>N_0(F)$.

These two conjectures lead to two others involving subordination and quasi-subordination.

CONJECTURE III (ROGOSINSKI, 1943). If $f(z)=\sum_1^\infty a_n z^n \prec F(z)=\sum_1^\infty A_n z^n$ in E where f and F are regular in E and F is univalent in E, then $|a_n|\leqq n|A_1|$, $n=1,2,\cdots$. The cases $n=1,2$ were shown by J. E. Littlewood [15]. The case $n=3$ now follows, since we have shown above that $|a_3|\leqq|D_1|^2+|D_3|^2+|D_5|^2\leqq 3|D_1|^2=3|A_1|$, not only in the subordination case but when $f\prec_q F$. Similarly, the case $n=4$ also follows from Friedland's result, mentioned earlier. W. Rogosinski [29] also showed that the conjecture was true for all n if F has real coefficients, or if F has complex coefficients and is starlike. The author [25] extended this result to the following. If $F(z)$ is close-to-convex in E, then for all $n|a_n|\leqq n|A_1|$ with equality only for the Koebe function for any $n>1$.

These three conjectures have remained unproven for long periods of time, for 52, 32 and 25 years respectively, so at my age it would probably do no harm to add another one here.

CONJECTURE IV (ROBERTSON, 1968). Let $f(z)=\sum_1^\infty a_n z^n \prec_q F(z)=\sum_1^\infty A_n z^n$ in E where $F(z)$ is univalent in E, then $|a_n|\leqq n|A_1|$, $n=1,2,3,\cdots$. Conjecture IV is true [27] for $n=1,2,3,4$ and for all sufficiently large values of $n>n_0(F)$. It is true for all n if the coefficients A_n are all real (a_n may be complex) [27]. It is true for all n if

$F(z)$ is starlike or spirallike in E [27].

Conjecture III, if true, implies the truth of the Bieberbach Conjecture I, since $F(z)$ is subordinate to itself. Conjecture IV, if true, implies the truth of Conjectures I and III, and is true if Conjecture II is true. Thus all of these conjectures are true if Conjecture II is true, namely, if $\sum_{k=1}^{\infty} D_{2k-1}z^{2k-1}$ represents a univalent function regular in E then

$$|D_1|^2 + |D_3|^2 + \cdots + |D_{2n-1}|^2 \leqq n|D_1|^2.$$

At this point I should like to mention briefly two other coefficient conjectures:

1. A. W. Goodman (1948) [9] conjectured that if $F(z) = \sum_1^{\infty} A_n z^n$ be regular and p-valent in E, i.e., if F takes on a value p-times and no value more than p-times in E, then for $n > p$

$$(2) \qquad |A_n| \leqq \sum_{k=1}^{p} \frac{2k(p+n)!}{(p+k)!(p-k)!(n-p-1)!(n^2-k^2)} |A_k|.$$

For $p = 1$ this reduces to $|A_n| \leqq n|A_1|$.

The conjecture is known to be true for a few special subclasses but unfortunately very little is known yet in this direction. Goodman and Robertson [10] showed that (2) holds for all $n > p$ whenever $f(z)$ is p-valently starlike in E and the coefficients A_k are all real. If the coefficients are complex the author [23] proved that it is true for each n in the case $p = 2$.

Recently A. E. Livingston [16] has shown that (2) is true for all n if $F(z)$ has the form $F(z) = A_{p-1}z^{p-1} + A_p z^p + \cdots$, $|z| < 1$, and $F(z)$ is p-valently close-to-convex in E. This is a great step forward in connection with an extremely difficult problem.

2. Another set of coefficient inequalities in the univalent $(p = 1)$ case is the set

$$(3) \qquad \big| n|A_n| - m|A_m| \big| \leqq |n^2 - m^2| \cdot |A_1|,$$

where $F(z) = \sum_1^{\infty} A_n z^n$ is regular and univalent in E, n and m being positive integers.

The inequality for $m = n - 1$ gives

$$n|A_n| \leqq (n-1)|A_{n-1}| + (2n-1)|A_1|$$

from which

$$|A_n| \leqq n|A_1|$$

follows by induction. The author [26] has proved that the inequalities (3) hold for each m, n whenever $F(z)$ maps E onto a domain comprised of parallel line segments, one segment on each line. (3) is also true [26] if F is merely close-to-convex in E provided $n-m$ is even. Both classes contain the Koebe function. (3) is also true [26] for $n=3$, $m=2$ for close-to-convex functions. In this case $|3|A_3|-2|A_2|| \leqq (3^2-2^2)\cdot|A_1|=5|A_1|$. However, (3) cannot be true in general for all F regular and univalent in E. J. A. Jenkins [12] showed recently that the sharp upper bound for $3|A_3|-2|A_2|$ is $(5.02\cdots)\cdot|A_1|$ when $F(z)$ is univalent in E. It is still an open question whether (3) is true for all close-to-convex functions if $n-m$ is not an even integer. The answer is an affirmative one if every starlike function $f(z) =z+\sum_2^\infty b_n z^n$, $|z|<1$, has coefficients which satisfy $|2b_n-b_2\cdot b_{n-1}| \leqq 2$, $n=1, 2, 3, \cdots$ $(b_0=0)$. These inequalities are true at least for $n=1$, 2 and 3.

References

1. L. Bieberbach, *Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln*, S.-B. Preuss. Akad. Wiss. **138** (1916), 940–955.

2. M. Biernacki, *Sur les fonctions univalentes*, Mathematica **12** (1936), 49–64.

3. N. G. de Bruijn, *Ein Satz über schlichte Funktionen*, Nederl. Akad. Wetensch. Proc. **44** (1941), 47–49. MR **2**, 274.

4. J. Dieudonné, *Sur les fonctions univalentes*, C. R. Acad. Sci. Paris **192** (1931), 1148–1150.

5. S. Friedland, *On a conjecture of Robertson*, Master of Science Dissertation, Technion-Israel Institute of Technology, Haifa, 1969.

6. P. R. Garabedian and M. Schiffer, *A proof of the Bieberbach conjecture for the fourth coefficient*. J. Rational Mech. Anal. **4** (1955), 427–465. MR **17**, 24.

7. G. M. Golusin, *Zur Theorie der schlichten Funktionen*, Mat. Sb. **6** (**48**) (1939), 383–388. MR **1**, 308.

8. ———, *On majorants of subordinate analytic functions*. I, Mat. Sb. **29** (**71**) (1951), 209–224. (Russian) MR **13**, 223.

9. A. W. Goodman, *On some determinants related to p-valent functions*, Trans. Amer. Math. Soc. **63** (1948), 175–192. MR **9**, 421.

10. A. W. Goodman and M. S. Robertson, *A class of multivalent functions*, Trans. Amer. Math. Soc. **70** (1951), 127–136. MR **12**, 691.

11. W. Hayman, *Multivalent functions*, Cambridge Univ. Press, London, 1968.

12. J. A. Jenkins, *On an inequality considered by Robertson*, Proc. Amer. Math. Soc. **19** (1968), 549–550. MR **37** #401.

13. Z. Lewandowski, *Starlike majorants and subordination*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **15** (1961), 79–84. MR **25** #2186.

14. ———, *Some remarks on a paper of M. S. Robertson*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **17** (1963), 43–46. MR **33** #5867.

15. J. E. Littlewood, *On inequalities in the theory of functions*, Proc. London Math. Soc. (2) (1925), 481–519.

16. A. E. Livingston, *The coefficients of multivalent close-to-convex functions*, Proc. Amer. Math. Soc. **21** (1969), 545–552.

17. K. Löwner, *Untersuchungen über schlichte konforme Abbildungen des Einheitskreises*, Math. Ann. **89** (1923), 103–121.

18. T. H. MacGregor, *Majorization by univalent functions*, Duke Math. J. **34** (1967), 95–102. MR **34** #6062.

19. R. N. Pederson, *A proof of the Bieberbach conjecture for the sixth coefficient*, Notices Amer. Math. Soc. **16** (1968), 181.

20. G. Pólya and I. J. Schoenberg, *Remarks on the de la Vallée Poussin means and convex conformal maps of the circle*, Pacific J. Math. **8** (1958), 295–334. MR **20** #7181.

21. M. O. Reade, *Sur une classe de fonctions univalentes*, C. R. Acad. Sci. Paris **239** (1954), 1758–1759, MR **16**, 579.

22. M. S. Robertson, *A remark on the odd schlicht functions*, Bull. Amer. Math. Soc. **42** (1936), 366–370.

23. ———, *Multivalently star-like functions*, Duke Math. J. **20** (1953), 539–549. MR **15**, 613.

24. ———, *Applications of the subordination principle to univalent functions*, Pacific J. Math. **11** (1961), 315–324. MR **23** #A1787.

25. ———, *The generalized Bieberbach conjecture for subordinate functions*, Michigan Math. J. **12** (1965), 421–429. MR **32** #2576.

26. ———, *A generalization of the Bieberbach coefficient problem for univalent functions*, Michigan Math. J. **13** (1966), 185–192. MR **33** #269.

27. ———, "Quasi-subordinate functions," in *Macintyre memorial volume*, Ohio Univ. Press, Athens, Ohio (to appear).

28. W. Rogosinski, *Über positive harmonische Entwicklungen und typische-reelle potenzreihen*, Math. Z. **35** (1932), 93–121.

29. ———, *On the coefficients of subordinate functions*, Proc. London Math. Soc. (2) **48** (1943), 48–82. MR **5**, 36.

30. Tao-shing Shah, *Goluzin's number* $(3-\sqrt{5})/2$ *is the radius of superiority in subordination*, Sci. Record **1** (1957), 219–222. MR **20** #6530.

31. L. Špaček, *Contribution à la théorie des fonctions univalentes*, Časopis Pěst. Mat.-Fys. **62** (1933), 12–19.

UNIVERSITY OF DELAWARE, NEWARK, DELAWARE 19711