COERCIVENESS OF THE NORMAL BOUNDARY PROBLEMS FOR AN ELLIPTIC OPERATOR

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Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^n \), with smooth boundary \( \Gamma \) (the theory is easily extended to compact manifolds). Let \( A \) be a differential operator of order \( 2m \) \((m \geq 1)\), with coefficients in \( C^\infty(\bar{\Omega}) \), such that \( A \) is uniformly strongly elliptic and formally self-adjoint in \( \bar{\Omega} \). We consider the \( L^2(\Omega) \)-realizations of \( A \), determined by boundary conditions of the form

\[
\gamma_j u - \sum_{k \in K, k < j} F_{jk} \gamma_k u = 0, \quad j \in J;
\]

here \( J \) and \( K \) are complementing subsets, each consisting of \( m \) elements, of the set \( M = \{0, \cdots, 2m-1\} \); the \( F_{jk} \) denote (pseudo-) differential operators in \( \Gamma \) of orders \( j-k \); and the \( \gamma_k \) denote the standard boundary operators: \( \gamma_0 u = u \mid \Gamma \), \( \gamma_k u = iD_n u \mid \Gamma \), for \( u \in C^\infty(\bar{\Omega}) \), where \( iD_n = \partial / \partial n \) is the interior normal derivative at \( \Gamma \). (1) is a reduced form of the usual normal type of boundary conditions, generalized to include pseudo-differential operators in \( \Gamma \).

Let \( \bar{A} \) be the operator in \( L^2(\Omega) \) defined by

\[
D(\bar{A}) = \{ u \in L^2(\Omega) \mid Au \in L^2(\Omega), u \text{ satisfies } (1) \},
\]

\[
\bar{A}u = Au \text{ on } D(\bar{A}).
\]

(The definition is given a sense by the general concept of boundary value introduced by Lions-Magenes [7]). We shall give below a necessary and sufficient condition on the operators \( F_{jk} \) (together with \( A \)) in order that \( A \) be \( m \)-coercive, i.e. satisfies

\[
\Re(\bar{A}u, u) + \lambda \| u \|_0^2 \geq c \| u \|_m^2, \quad \forall u \in D(\bar{A}),
\]

for some \( \varepsilon > 0, \lambda \in \mathbb{R} \). The condition has two parts:

1° it is necessary that the \( F_{jk} \) with \( j \) and \( k \geq m \) are certain functions of the \( F_{jk} \) with \( j \) and \( k < m \) in order that \( \bar{A} \) be even lower bounded (Theorem 1);

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\(1 \) Here \( \| u \|_s \) denotes the norm in the Sobolev space \( H^s(\Omega) \), \( s \in \mathbb{R} \).
2° when 1° is fulfilled, the m-coerciveness is equivalent with an algebraic condition on the principal symbols (Theorem 2).

Theorems 1–2 arise as corollaries of a general result (Theorem 3), which permits application of [4], [5].

In [1], Agmon gave an algebraic condition for m-coerciveness of self-adjoint realizations defined by differential boundary operators; restricted to such realizations, our condition is equivalent with his. Our result also extends those of Fujiwara-Shimakura [3] and Grubb [5], treating certain nonselfadjoint classes of (1). The theory avoids the classical considerations of integro-differential forms, which are not very convenient for the question of necessity. However, our m-coercive $A$ are variational in the sense of [5] (i.e., $A + \lambda$ is regularly accretive in Kato’s sense, for suitable $\lambda \in \mathbb{R}$).

1. A necessary condition for lower boundedness. For a $(p \times q)$

$$E = ((E_{jk})_{j=0, \ldots, p-1, \atop k=0, \ldots, q-1}$$

and two ordered subsets $N_1$ and $N_2$ of $\{0, \ldots, p-1\}$ resp. $\{0, \ldots, q-1\}$, we denote the minor $((E_{jk})_{j\in N_1, k\in N_2})$ by $E_{N_1,N_2}$.

Similarly for a row- or column-vector $\phi = \{\phi_0, \ldots, \phi_{p-1}\}$ we denote $\{\phi_j\}_{j\in N_1}$ by $\phi_{N_1}$. We also use $\phi_{N_1}$ to indicate a vector $\{\phi_j\}_{j\in N_1}$ indexed by $N_1$.

Let $J$, $K$ and $M$ be as above, then we introduce the ordered subsets of $M$: $M_0 = \{0, \ldots, m-1\}$, $M_1 = \{m, \ldots, 2m-1\}$, $J_0 = J \cap M_0$, $J_1 = J \cap M_1$, $K_0 = K \cap M_0$ and $K_1 = K \cap M_1$. When $N \subseteq M$ we set $N' = \{n | 2m-1 - n \in N\}$, considered again as an ordered subset of $M$.

The “Cauchy” boundary operator $\{\gamma_0, \ldots, \gamma_{2m-1}\}$ will be denoted by $\rho$.

With this notation, (1) is equivalent with

$$\rho_{\phi, \psi} = F_0 \rho_{\psi, \phi_0} \psi + F_1 \rho_{\psi, \phi_1} \psi + F_2 \rho_{\psi, \phi_2} \psi,$$

where $F_0$, $F_1$ and $F_2$ are the matrices of (pseudo-)differential operators (where we put $F_{jk} = 0$ for $j \leq k$): $F_0 = ((F_{jk})_{j\in J_0, k\in K_0}$, $F_1 = ((F_{jk})_{j\in J_1, k\in K_0}$ and $F_2 = ((F_{jk})_{j\in J_1, k\in K_1}$.

(Evident modifications when empty index sets occur.) They are of types $(\psi, j)_{j\in J_0, k\in K_0}$, $(\psi, j)_{j\in J_1, k\in K_0}$ and $(\psi, j)_{j\in J_1, k\in K_1}$, respectively. (The notion of type is a convenient generalization of order to matrices, the principal symbol $\sigma^0$ is defined accordingly, see Hörmander [6], or [5].) Note the way in which $F_0$ and $F_2$ are minors of matrices with zeroes in and above the diagonal; we shall say that they are subtriangular.

The operator $F_0$, which maps $\prod_{k \in K_0} H^{t-k}(\Gamma)$ into $\prod_{k \in J_0} H^{t-k}(\Gamma)$,
all \( s \in R \), can be supplemented with the identity on \( \prod_{k \in K_0} H^{s-k}(\Gamma) \) to yield an operator \( \Phi \) from \( \prod_{k \in K_0} H^{s-k}(\Gamma) \) to \( \prod_{k \in M_0} H^{s-k}(\Gamma) \):

\[
\Phi: \phi_{K_0} \mapsto \psi_{M_0}, \quad \text{where } \psi_{K_0} = \phi_{K_0}, \; \psi_{J_0} = F_0 \phi_{K_0}.
\]

We write in short

\[
\Phi = \begin{pmatrix} I_{K_0} \\ F_0 \end{pmatrix}, \quad \text{where } I_{K_0} = ((\delta_{jk}))_{j,k \in K_0}.
\]

The adjoint \( \Phi^* \) sends \( \phi_{M_0} \) into \( \phi_{K_0} + F_0 \phi_{J_0} \) and is written in short as \( \Phi^* = (I_{K_0} F_0^\circ) \). \( \Phi \) and \( \Phi^* \) are (pseudo-)differential operators of types \( (-k, -j) \) respectively \( (k, j) \) with analogical notation for their symbols one has e.g. \( \sigma^0(\Phi^*) = (I_{K_0} - \sigma^0(F_0^\circ)) \).

At the points of \( \Gamma \) one may write \( A \) in normal and tangential coordinates

\[
A = \sum_{l=0}^{2m} A_l D_n^l,
\]

where the \( A_l \) denote differential operators in \( \Gamma \) of orders \( 2m - l \); \( A_{2m} \) is a positive function. Then one has the Green's formula

\[
(Au, v) - (u, Av) = \int_\Gamma \alpha p u \cdot \nu \, d\sigma, \quad u, v \in C^\infty(\Omega),
\]

where \( \alpha \) is a \( (2m \times 2m) \)-matrix of differential operators in \( \Gamma \): \( \alpha = ((\alpha_{jk}))_{j,k \in M} \) where each \( \alpha_{jk} \) has the form \( iA_{j+k+1} + \) differential orders less than \( 2m - (j+k+1) \) (we put \( A_l = 0 \) for \( l > 2m \)), cf. Seeley [8], or [5]. We note that \( \alpha^* = -\alpha \), and that \( \alpha \) is skew-triangular and invertible with \( \alpha^{-1} \) a differential operator; \( \alpha \) is elliptic of type \( (-k, -2m+j+1) \) for some \( j \).

**Theorem 1.** If \( \bar{A} \) is lower bounded, that is, if there exists \( \lambda \in R \) such that \( \Re(\bar{A} u, u) \geq \lambda \| u \|_{F_0}^2 \), \( \forall u \in D(\bar{A}) \) then \( K_0 = J_1' \), and

\[
F_2 = - (\Phi^* \alpha_{M_0 J_1})^{-1} \Phi^* \alpha_{M_0 K_1}.
\]

(Here \( \Phi^* \alpha_{M_0 J_1} \) is invertible when \( K_0 = J_1' \), thanks to the special character of \( \alpha \) and the subtriangularity of \( F_0 \).)

**Remark 1.** The case treated by Fujiwara-Shimakura [3], Fujiwara [2] and Grubb [5, 4.3–4.4] is the case where

\[
K_0 = J_1' = \{ m - p, \ldots, m - 1 \}
\]

for some \( p \leq m \), here \( F_0 \) and \( F_2 \) are 0 by their subtriangularity; the case in Grubb [5, 4.5] takes general \( K_0 \) but \( F_0 = 0 \).
2. The condition for $m$-coerciveness. In accordance with (6), the principal symbol of $A$ may at points $y \in \Gamma$ be written in the form 
$$a(y, \eta, \tau) = \sum_{i=0}^{2m} a^i(y, \eta) \tau^i,$$
where $a_i(y, \eta)$ denotes the principal symbol of $A_i$; here $\eta$ belongs to the fibre at $y$ of the cotangent bundle $T^*(\Gamma)$, and $\tau \in \mathbb{R}$. For each $(y, \eta)$ with $\eta \neq 0$, the polynomial $a(y, \eta, \tau)$ has exactly $m$ roots $\{\tau_i^+(y, \eta)\}_{i=1}^{m}$ in \{\lambda \in \mathbb{C} \mid \text{Im} \lambda > 0\}. We can then form the polynomial $\prod_{i=1}^{m} (\tau - \tau_i^+(y, \eta)) = \sum_{i=0}^{m} s_i(y, \eta) \tau^i$, and use the coefficients to define the following $(m \times m)$-matrix valued functions on the nonzero subbundle $T^*_y(\Gamma)$ of $T^*(\Gamma)$:
$$S_i(y, \eta) = (((s^i(y, \eta)))_{j,k \in \mathbb{M}_s})$$
and
$$S_m(y, \eta) = (((s^m_{j,\eta-1}+i\chi))_{j,k \in \mathbb{M}_s},$$
where we put $S_i = 0$ for $i \notin \{0, \ldots, m\}$.

Denoting by $\Pi^*$ the skew-unit matrix $((\delta_{i,m-1-k}))_{j,k \in \mathbb{M}_s}$, we finally introduce
$$Q = i\Pi^* S_m S_m, R = i\Pi^* S_m S_0,$$
and $Q$ denotes the complex conjugate of $S$. (More details in [5, Chapter 4], in fact $Q = A_{2m}^{-1} \sigma^0(\mathbb{A}_s \mathbb{M}_s)$, and $R$ is the principal symbol of a certain pseudo-differential operator in $\Gamma$.)

**Theorem 2.** $\tilde{A}$ is $m$-coercive if and only if it satisfies (i) and (ii):

(i) $K_0 = J_1^*$, and $F_2 = -((\Phi^* A_{M_0})^{-1} \Phi^* A_{M_0})^*$,

(ii) Let $J_2 = \{j \mid j+m \in J_1\}$, and let $E(y, \eta)$ be the matrix valued function on $T^*_y(\Gamma)$:

$$E = \sigma^0(\Phi^*) Q_{M_0 J_2} \sigma^0(F_1) + \sigma^0(\Phi^*) R \sigma^0(\Phi),$$

then $E + E^*$ is positive definite on $T^*_y(\Gamma)$.

In the affirmative case, $\tilde{A}$ is $2m$-regular ($\tilde{A} u \in H^s(\Omega) \Rightarrow u \in H^{s+2m}(\Omega), \forall s \geq 0$), and $\tilde{A}^*$ is also $m$-coercive and $2m$-regular.

3. Explanations and further developments. The first step in our proof of Theorems 1–2 is the transformation of (4) into an equivalent boundary condition of the form

$$\gamma J_0 u = F_0 \gamma K_0 u, \quad \chi J_1 u = G_1 \gamma K_0 u + G_2 \chi K_1 u,$$

where $\gamma$ and $\chi$ denote the $m$-vectors of boundary operators: $\gamma = \rho M_0, \chi = \rho M_0 M_1 + \frac{1}{2} \rho M_0 M_1 M_0^*$, with which Green's formula (7) takes the simple form: $(Au, v) - (u, Av) = \int \chi u \cdot \bar{\gamma} v - \gamma u \cdot \bar{\chi} v) d\sigma.$ Note that $\chi = \{\chi_k\}_{k \in \mathbb{M}_s}$, where $\chi_k$ is of order $2m-k-1$. There is 1-1 correspondence between the systems $(F_0, F_1, F_2)$ and $(F_0, G_1, G_2)$ (we omit the formulae); $G_2^*$ is again subtriangular.

Assuming, as we may, that the Dirichlet problem for $A$ is uniquely solvable, we define the operator $P_{\gamma, x}$ in $\mathcal{D}(\Gamma)^m$ by: $P_{\gamma, x} \phi = \chi z$, where $z$ is the solution of $Az = 0$ in $\Omega, \gamma z = \phi$ (cf. [4], [5]). $P_{\gamma, x}$ is self-adjoint pseudo-differential operator in $\Gamma$ of type $(-k, -2m+j+1)_{j,k \in \mathbb{M}_s}$ (Vainberg-Grušin [9]); its principal symbol is described in detail in [5, Chapter 4].
THEOREM 3. In addition to the notations introduced above, let $\Psi$ be the operator analogous to $\Phi$ with $F_0$ replaced by $-G^*$.

Let $X = \Phi(\bigcap_{k \in K_0} H^{-k - 1/2}(\Gamma))$ and let $Y = \Psi(\bigcap_{k \in K_0} H^{-k - 1/2}(\Gamma))$. Let $\Phi_1$ and $\Psi_1$ be the restrictions of $\Phi$ and $\Psi$ with domains $\bigcap_{k \in K_0} H^{-k - 1/2}(\Gamma)$ respectively, and ranges $X$ resp. $Y$, clearly they are isomorphisms.

Finally, introduce the pseudo-differential operator $\mathcal{L}_1$ of type $(-k, -2m+j+1)_{j \in J_1, k \in K_0}$:

$$\mathcal{L}_1 = G_1 - \Psi^* P_{\gamma, \Phi}. $$

Then $\tilde{A}$ corresponds, in the sense of [4, Theorem III 2.1] (based on the Dirichlet problem), to the operator $L: X \to Y'$ defined by

$$D(L) = \left\{ \phi \in X \mid \mathcal{L}_1 \Phi_1^{-1} \phi \in \prod_{k \in J_1'} H^{k+1/2}(\Gamma) \right\},$$

$$L\phi = (\Psi_1)^{-1} \mathcal{L}_1 \Phi_1^{-1} \phi, \quad \text{when } \phi \in D(L).$$

Theorem 1 follows from this by use of [4, Theorem III 4.3]: Lower boundedness of $\tilde{A}$ implies $X \subset Y$, and then by the subtriangle property $\Phi = \Psi$, so that $K_0 = J_1'$ and $F_0 = -G^*$, which leads to (8). Note that then $X = Y$.

Theorem 2 uses [5, Corollary 2.4]: $\tilde{A}$ is $m$-coercive if and only if $L$ is $m$-coercive, i.e., $X \subset Y$ and $\exists c > 0, \lambda \in \mathbb{R}$ so that

$$\text{Re}(L\phi, \phi) + \lambda \|\phi\|^2_{L^{-1/2}} \geq c \|\phi\|^2_{L^{m-1/2}} \text{ on } D(L).$$

This is equivalent with a similar property for $L_1 = \Psi^* L \Phi_1$, which is a certain "realization" of $L_1$, and here the property amounts (besides $\Phi = \Psi$) to the positive definiteness of $\sigma^0(\mathcal{L}_1 + \Phi_1^*)$ (in fact $E = A_{2m}^{-1} \sigma^0(\mathcal{L}_1)$); the computations resemble those in [5]. The last statement in Theorem 2 uses the ellipticity of $L_1$.

REMARK 2. The self adjoint $m$-coercive $\tilde{A}$ are characterized by Theorem 2 (i), (ii), plus selfadjointness of $G_1 = \Phi^* G_0 \Phi + \Phi^* \Phi^* M_0 \Phi$ (then $E$ is also selfadjoint).

REMARK 3. Theorem 3 gives a basis for the discussion of many other properties of $\tilde{A}$, because of the way in which they are preserved by the correspondence between $\tilde{A}$ and $L$, see [4], [5]. Regarding coerciveness, we mention that:

1° the conditions in Theorem 2 are also necessary and sufficient for $(m-\varepsilon)$-coerciveness with $\varepsilon \in [0, 1/2]$ [cf. Fujiwara-Shimakura [3]],

2° the discussion of $(m-1/2)$-coerciveness in Fujiwara [2] (related to subellipticity [6]) seems extendable to the present case.

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$\Box_{k \in M_0} H^{-k - 1/2}(\Gamma)$ denotes the norm in $\Pi_{k \in M_0} H^{-k - 1/2}(\Gamma)$.
3° a necessary condition for lower boundedness ("0-coerciveness") is the positive semidefiniteness of $\sigma(\mathcal{L}_1 + \mathcal{L}_1^*)$ (cf. [5, Theorem 4.3]). Let us mention that lower boundedness + 2m-regularity do not imply m-coerciveness as in the selfadjoint case; examples using pseudo-differential operators: take $\mathcal{L}_1$ elliptic with $\mathcal{L}_1^* = - \mathcal{L}_1$.

Concerning extensions of the results to operators $A$ that are merely strongly elliptic, let us mention that the case $K_0 = J'_1 = \{ m - \rho, \ldots, m - 1 \}$ has been treated by Fujiwara [2]; the device of [2] does not extend to our general case.

References


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