ORDER ALGEBRAS

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A partially ordered set $P$ in which every pair of elements has a greatest lower bound is a semigroup, with $pq = p \wedge q$, and hence is naturally associated with a semigroup algebra $\mathbb{Z}[P]$ over the integers. For finite $P$ Solomon has given [3] a marvelously ingenious construction of an analogous sort of algebra even when $P$ is not a semilattice and so cannot be made into a semigroup. Semigroup algebras and Solomon's "Möbius algebras" have applications in combinatorial problems involving the underlying orders.

Now in a recent study [2] of valuations and Euler characteristics on lattices Rota introduced an ostensibly quite different sort of algebra he called a "valuation ring" which, rather surprisingly, plays a role like that of a semigroup algebra. More surprising, in view of their entirely different genesis and description, is that Rota's valuation ring can be shown to include Solomon's Möbius algebra as a special case.

Rota's construction, when used to associate such an algebra to a partial order $P$ (which is only one outgrowth of his inquiry), leads in stages through several different structures. The results implicitly provide a recursive procedure for computing products in the valuation ring $V(P)$, but give no direct formula. Solomon, on the other hand, defined his Möbius algebra by giving an explicit, if rather complicated, formula to express products of elements of $P$ as linear combinations of $P$-elements. The purpose of this note is to determine from Rota's construction an explicit formula for products in $V(P)$ which depends only on the order structure of $P$. This will show at once that Rota's construction includes Solomon's, and it can be recast in a particularly simple form that clarifies further consequences and applications.

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1. **The Rota construction.** Let \( L = \{S, T, \cdots \} \) be any distributive lattice under \( \cup \) and \( \cap \), made into a semigroup by setting \( ST = S \cap T \). In the semigroup algebra \( K[L] \) over a commutative ring \( K \) the submodule \( Q \) generated by all \( S + T - ST - S \cup T \) with \( S \) and \( T \) in \( L \) is an ideal. Since valuations on \( L \) are just those functionals which are identically zero on \( Q \), Rota calls the quotient \( K[L]/Q \) the **valuation ring** \( V(L, K) \). The special case of interest in this note has \( L \) the lattice of "order ideals" of a partial order \((P, \leq)\) and \( K = \mathbb{Z} \), the ring of integers.

Let \( P \) be such that every cone \( C_p = \{ q \in P : q \leq p \} \) is finite and define \( L \) to be the ring of sets generated by all cones, with \( \emptyset \) added. Then \( L \) is a distributive lattice whose finite elements admit the convenient height function, \( ht S = |S| \) (number of elements in \( S \)). The quotient \( Z[L]/Q \) may in this case, because of its likeness to the semigroup algebra of a semigroup, be called the **order algebra**, \( V(P) \), of \( P \).

Identifying elements of \( L \) with their images in \( V \), Rota extends the identity defining \( Q \) to give a general inclusion-exclusion formula that expresses any finite union of lattice elements as a linear combination:

\[
S_1 \cup \cdots \cup S_r = \sum_{i=1}^r S_i - \sum_{i<j} S_i S_j + \sum_{i<j<k} S_i S_j S_k - \cdots .
\]

**Lemma 1.** Any \( S \) of finite height in \( L \) is a well-defined linear combination, \( S = \sum_{p \in P} \phi_S(p) C_p \), of cones contained in \( S \): that is \( \phi_S(p) = 0 \) unless \( C_p \subset S \).

The proof is by induction on the height of \( S \). If \( ht S = 0 \) then \( S = \emptyset \) and this is \( T + T - TT - T \cup T = 0 \).

Any other element of finite height in \( L \) is either a cone or a finite union of join irreducibles (i.e. cones) of finite height. Now assume the lemma for elements of height \( < h \) and suppose \( ht S = h \). If \( S \) is not itself a cone it must be an irredundant union, \( S = C_{p_1} \cup \cdots \cup C_{p_r} \), of the maximal cones contained in \( S \). By (\*), \( S = \sum C_{p_1} - \sum C_{p_i} C_{p_j} + \cdots \), where each term \( C_{p_1} \cdots C_{p_k} \) on the right has height \( < h \) and hence, by induction, is a well-defined linear combination of cones \( C_q \) contained in it, and thus in each \( C_{p_i} \). Then \( S \) is the well-defined linear combination gotten by adding all such terms, and furthermore each \( C_q \subset C_{p_i} \subset S \).

(If \( S \) is written as a nonirredundant union of cones, which can only be done by using all the maximal \( C_{p_i} \) in \( S \) and other cones \( C_r \) contained within some of them, it is easy to show that the added contribution from the \( C_r \)'s amounts to zero.)

Thus \( V \) essentially consists of all linear combinations of cones and
its multiplication can be taken to define a (commutative $\mathbb{Z}$-algebra) product among $P$-elements, say $\circ$, by the rule: $x \circ y = \sum_{p \in P} \phi_{xy}(p) \cdot p$ if and only if $C_x C_y = \sum_{p \in P} \phi_{xy}(p) C_p$. This way of writing the $V(P)$ product brings out the analogy with semigroup algebras; of course, $V(P)$ is the integral semigroup algebra on $P$ if and only if $P$ is a semilattice.

2. Explicit formula for the product. Rota’s procedures show how to compute such products by working upward from minimal elements, but provide no direct way to determine $C_x C_y$. With only these recursive techniques to build on it is natural to seek an explicit formula by repeated use of induction in the identity (*).

Suppose now that the maximal cones in a given $C_x C_y = C_x \cap C_y$ are $C_{p_1}, \ldots, C_{p_r}$. Then the expansion (*) can be rewritten as

$$C_x C_y = C_{p_1} \cup \cdots \cup C_{p_r} = \sum_{i=1}^{r} C_{p_i} - \sum_{i<j} C_{p_i} C_{p_j} + \sum_{i<j<k} C_{p_i} C_{p_j} C_{p_k} - \cdots.$$

Determining any coefficient $\phi_{xy}(p)$ calls for further expanding each term on the right that is not already a cone until ultimately every term is reduced to a linear combination of cones, and then adding over all terms.

In fact, however, it is simpler to determine first the sum of all $\phi_{xy}(q)$ for $q$ in the filter $F_p = \{q \in P : q \geq p\}$ above $p$. Suppose $C_{q_1} \cdots C_{q_t}$ is any term sooner or later arising in the expansion of (**), and that its expression as a linear combination of cones is $\sum \pi_r C_r$. Then the sum of all those $\pi_r$ for which $r \in F_p$ can be described as the “contribution” of the term $C_{q_1} \cdots C_{q_t}$ to the sum

$$\sigma_{sy}(p) = \sum_{q \in F_p} \phi_{xy}(q).$$

**Lemma 2.** If $C_{q_1}, \ldots, C_{q_t}$ are cones within $C_x \cap C_y$ then:

(a) if there is any $i$ with $p \leq q_i$ the contribution of $C_{q_1} \cdots C_{q_t}$ to $\sigma_{xy}(p)$ is 0;

(b) if $p \leq q_i$ for each $i$ this contribution is 1.

**Proof.** If again $C_{q_1} \cdots C_{q_t} = \sum \pi_r C_r$ then whenever $\pi_r \neq 0$ for some $r \in F_p$ it must be that $p \leq r \leq q_i$ for each $i$.

The proof of (b) is by induction on $h$, the maximum of the heights $ht(C_p, C_q)$ from $C_p$ to $C_q$. For $h = 0$ the term $C_{q_1} \cdots C_{q_t} = C_p$ does contribute 1 to $\sigma_{xy}(p)$. Assume the lemma true whenever the maximum of these heights is less than $h$ and now suppose that $q \leq p_i$ for
each \( i \) and max \( \text{ht}(C_p, C_q) = h \). Notice that if \( t = 1 \) the term is just \( C_p \), and hence does contribute 1 to the sum.

Now with \( t > 1 \) and all \( \text{ht}(C_p, C_q) \leq h \) any cone \( C_r \) which is maximal in \( C_q \cap \cdots \cap C_q \) must have \( \text{ht}(C_p, C_r) < h \) so that for

\[
C_q \cdots C_q = C_{r_1} \cup \cdots \cup C_{r_t} = \sum_{i=1}^{t} C_{r_i} - \sum_{i<j} C_{r_i} C_{r_j} + \cdots
\]

each term on the right contributes 1 to \( \sigma_{xy}(p) \), by induction, and hence the total contribution to the sum from \( C_{q_1} \cdots C_{q_t} \) is just

\[
\binom{t}{1} - \binom{t}{2} + \binom{t}{3} - \cdots + (-1)^{t-1} \binom{t}{t} = 1.
\]

**Theorem.** For any \( x, y \) and each \( p \in C_x \cap C_y \) the sum \( \sigma_{xy}(p) = 1 \).

**Proof.** Suppose \( C_{p_1}, \ldots, C_{p_s} \) are the maximal cones in \( C_x \cup C_y \) with subscripts so chosen that the first \( s \) generators \( p_1, \ldots, p_s \) are in the filter \( F_p \) and the rest are not. The terms \( C_{p_1} \cdots C_{p_i} \) of the expansion (**) can be split into two classes according as all \( p_i \in F_p \) or not. Then (**) gives \( C_x C_y = \sum' + \sum'' \) where each term in the former sum (\( \sum' \)) has all \( p_i \in F_p \) and each term in the latter has at least one \( p_i \notin F_p \). Now Lemma 2 shows

(a) that the whole contribution to \( \sigma_{xy}(p) \) comes from the first sum (\( \sum' \)) and

(b) that each term in this sum contributes 1. But \( \sum' \) is precisely the same as the expansion by (*) of \( C_{p_1} \cup \cdots \cup C_{p_s} \) and hence

\[
\sigma_{xy}(p) = \binom{s}{1} - \binom{s}{2} + \cdots + (-1)^{s-1} \binom{s}{s} = 1.
\]

A straightforward Möbius inversion using the \( \mu \)-function of \( P \) (see [1]) now yields a simple formula for \( \phi_{xy}(p) \).

**Corollary.** For each \( x, y \) and \( p \) in \( P : \phi_{xy}(p) = \sum_{q \in P} \mu(p, q) \sigma_{xq}(q) \).

Hence the product, \( \circ \), defined by cone multiplication is given by \( x \circ y = \sum_{p \in P} (\sum_{q \in C_x \cap C_y} \mu(p, q)) \cdot p \).

The product takes this form since \( \sigma_{xq}(q) = 1 \) or 0 according as \( q \in C_x \cap C_y \) or not.

When the order on \( P = \{ x_0, x_1, x_2, \cdots \} \) can be extended to that of the natural numbers its incidence algebra \( \alpha(P) \) (see [1]) can be taken to be upper triangular matrices including the Möbius function \( M \) with \( m_{ij} = \mu(x_i, x_j) \) and its inverse the zeta function \( Z(z) = 1 \) or 0 as \( x_i \leq x_j \) or not. Representing each \( x_i \in P \) by the column vector with
ith component 1 and all others 0 makes \( V(P) \) a left \( \alpha(P) \) = module consisting of finitely nonzero vectors \( x = \sum \xi_i x_i \) and having a convolution, \( \ast \), given by \( (\sum \xi_i x_i) \ast (\sum \eta_j x_j) = \sum_i \sum_j \xi_i \eta_j (x_i \circ x_j) \).

**Corollary.** If \( \cdot \) denotes componentwise multiplication of column vectors, then the product of \( P \)-elements is given by \( x_i \circ x_j = M(Zx_i \cdot Zx_j) \) and hence the convolution \( x \ast y = M(Zx \cdot Zy) \).

Thus the operator \( Z \) defines a convolution transform \( Z(x \ast y) = Zx \cdot Zy \), and this extends to order algebras the interesting concepts and applications introduced by Tainiter [4] for finite semigroups.

**References**


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