ON BIEBERBACH EILENBERG FUNCTIONS

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I. Introduction. In this paper we bring the following two results:

Suppose that \( F(z) = b_1 z + b_2 z^2 + \cdots \) is a B.E. function (i.e. \( F(z) \) is regular in the unit circle, \( F(z)F(\zeta) \neq 1 \) for any \( |z|, |\zeta| < 1 \) and \( F(0) = 0 \)). Then we have

\[
\sum_{k=1}^{\infty} |b_k|^2 \leq 1.
\]

This result contains, of course, the result

\[
|b_n| \leq 1, \quad n = 1, 2, \cdots
\]

which was conjectured by Rogosinsky [8] and was solved about ten years later by Lebedev and Milin [5].

The second result deals with univalent B.E. function \( F(z) = b_1 z + b_2 z^2 + \cdots \). For such function we have the following

\[
|b_n| \leq e^{-c/2}(n - 1)^{-1/2}, \quad n = 2, 3, \cdots,
\]

where \( c \) is Euler constant.

This result is sharp in order of magnitude and the constant cannot be improved to be better than \( e^{-1/2} \).

II. The results of Lebedev and Milin. Lebedev and Milin found [6], [7] some important results concerning coefficients of exponential functions which we quote here.

**Lemma 1.** Let \( A_1, A_2, A_3, \cdots \) be an infinite sequence of arbitrary complex numbers such that \( \sum_{k=1}^{\infty} k|A_k|^2 < \infty \). Then for \( \exp \sum_{k=1}^{\infty} A_k z^k = \sum_{k=0}^{\infty} D_k z^k \) we have

\[
\sum_{k=0}^{\infty} |D_k|^2 \leq \exp \sum_{k=1}^{\infty} k|A_k|^2
\]

with equality only in the case \( A_k = \rho^k \eta^k / k, k = 1, 2, \cdots \) where \( 0 \leq \rho < 1 \) and \( |\eta| = 1 \).

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Lemma 2. Let \( \{A_k\} \) and \( \{D_k\} \) be defined as in Lemma 1 (without the limitation \( \sum_{k=1}^{\infty} k |A_k|^2 < \infty \)). Then

\[
|D_n|^2 \leq \exp \left( \sum_{k=1}^{n} k |A_k|^2 - \sum_{k=1}^{n} 1/k \right), \quad n = 1, 2, \ldots
\]

with equality only in the case \( A_k = \eta k/k \) for \( k = 1, 2, \ldots, n \) and \( |\eta| = 1 \).

III. Schiffer-Garabedian inequalities. We quote here a theorem of Garabedian and Schiffer [1]:

Lemma 3. Suppose that \( F(z) \) is a univalent B.E. function. Then we have for

\[
\log \frac{F(z) - F(\zeta)}{(z - \zeta)[1 - F(z)F(\zeta)]} = \sum_{n,m=0}^{\infty} \gamma_{nm} a_{n-m},
\]

\[
\text{Re} \left\{ \sum_{n,m=0}^{N} \lambda_n \lambda_m \gamma_{nm} \right\} \leq \sum_{n=1}^{N} \frac{|\lambda_n|^2}{n}
\]

where \( \lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_N \) is a finite sequence of complex constants with \( \lambda_0 \) real.

This remarkable result was proved first in [1] by variational methods. Later the result was proved in [3] by area methods. We note that in [1] the result was formulated in a different manner.

IV. Coefficients of B.E. functions. From Lemma 3 we deduce immediately the following:

\[
\sum_{k=0}^{\infty} k |\gamma_0|^2 \leq \log \frac{1}{|F'(0)|^2}.
\]

(Indeed from Lemma 3 we have

\[
\lambda_0^2 \text{Re}\{\log F'(0)\} + 2\lambda_0 \text{Re} \left\{ \sum_{n=1}^{N} \lambda_n \gamma_{n0} \right\} \leq 2 \sum_{n=1}^{N} \frac{|\lambda_n|^2}{n}.
\]

By substitution \( \lambda_0 = 2, \lambda_n = n \gamma_{n0} \) we get (8).)

We are now in a position to prove

Theorem 1. Let \( F(z) = b_1z + b_2z^2 + \cdots \) be a B.E. function; then (1) follows.

Proof. By substituting \( \zeta = 0 \) in (6) we have

\[
\log \frac{F(z)}{z} = \sum_{n=0}^{\infty} \gamma_{n0} z^n, \quad \frac{F(z)}{zF'(0)} = \exp \left( \sum_{k=1}^{\infty} \gamma_{0k} z^k \right) = \sum_{k=1}^{\infty} \frac{b_k}{F'(0)} z^{k-1}.
\]
By Lemma 1 and (8) we get

$$
\sum_{k=1}^{\infty} \frac{|b_k|^2}{|F'(0)|^2} \leq \exp \left( \sum_{k=1}^{\infty} k |\gamma_{ok}|^2 \right) \leq \frac{1}{|F'(0)|^2}.
$$

So our theorem follows for univalent B.E. function. The result is generalized to the general class by the principle of subordination [2, pp. 424-425], [9].

**Remark 1.** The result is sharp for the B.E. function $F(z) = z^n$, $n = 1, 2, \cdots$ and also for Jenkins’s functions [4]

$$
F(z) = \frac{(1 - r^2)^{1/2}z}{1 + r^2z}, \quad 0 \leq r < 1.
$$

**Remark 2.** Jenkins’s result [4]

$$
F(z) \leq \left| z \right| / (1 - \left| z \right|^2)^{1/2}
$$

follows easily from Theorem 1.

**Theorem 2.** Let $F(z) = b_1z + b_2z^2 + \cdots$ be a univalent B.E. function. Then we have

$$
|b_n| < e^{-\epsilon/2(n - 1)^{-1/2}}, \quad n = 2, 3, \cdots
$$

where $c$ is Euler constant.

**Proof.** By Lemma 2 and (8), (9) we have

$$
\frac{|b_n|^2}{|F'(0)|^2} \leq \exp \left( \sum_{k=1}^{n-1} k |\gamma_{ok}|^2 - \sum_{k=1}^{n-1} 1/k \right) \leq \frac{\exp \left( - \sum_{k=1}^{n-1} 1/k \right)}{|F'(0)|^2}
$$

So $|b_n|^2 < e^{-\epsilon(n - 1)^{-1}}$ which is another form of our theorem. For Jenkins’s functions (11) we have $|b_n|^2 = (1 - r^2)^{2(n-1)}$. If $1 - r^2 = 1/(n - 1)$ we have

$$
|b_n|^2 = \frac{1}{n - 1} \left( 1 - \frac{1}{n - 1} \right)^{n-1} \sim \frac{1}{e(n - 1)}.
$$

So the order of magnitude is the best possible and the argument for the constant also follows.

**References**


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