The purpose of this note is to indicate how certain asymptotic methods developed for ordinary differential equations can be extended and applied to initial-boundary value problems for nonlinear parabolic and hyperbolic equations. This is done by considering the initial-boundary value problem as a Cauchy problem for an ordinary differential equation in an abstract space.

We consider the initial value problem

\[ \varepsilon \frac{dv}{dt} - A(t, \varepsilon)v = f(t, v, \varepsilon), \quad 0 \leq t \leq T, \quad v(0) = \hat{v}(\varepsilon) \]

where \( v \) is an element of a Banach space \( E \) and \( \varepsilon > 0 \) is a small parameter. The (possibly unbounded) linear operators \( A \) are assumed to have a common domain of definition \( \mathcal{D} \) independent of \( (t, \varepsilon) \), and the function \( f \) is assumed to have continuous derivatives with respect to \( t, \varepsilon \) and continuous Fréchet derivatives with respect to \( v \). Finally, \( \hat{v}(\varepsilon) \in \mathcal{D} \) has continuous derivatives with respect to \( \varepsilon \).

We will outline here a method for finding an expansion for the solution of (1) which is valid as \( \varepsilon \to 0 \).

1. **Formal method.** We begin by formally describing the procedure. These steps will be justified by Theorems 1–3. Suppose

(I) the operator \( A(t, \varepsilon) \) has a bounded inverse for each \( (t, \varepsilon) \) and \( A(t, \varepsilon)A^{-1}(0, 0) \) has continuous derivatives with respect to \( (t, \varepsilon) \).

Assuming for the moment that (1) has a solution for \( \varepsilon > 0 \), we differentiate (1) successively with respect to \( \varepsilon \) and set \( \varepsilon = 0 \) in the results. This gives the system of equations

\[
\begin{align*}
(2a) \quad & -A(t, 0)v_0 = f(t, v_0, 0) \\
(2b) \quad & -[A(t, 0) + f_r(t, v_0(t), 0)]v_r = R_r(t), \quad r = 1, 2, \ldots,
\end{align*}
\]

for the coefficients \( v_r \) of the Taylor expansion of \( v \) about \( \varepsilon = 0 \). Next, we make the change of variables \( t = \epsilon \tau \) in (1):

\[ dV/d\tau - A(\epsilon \tau, \epsilon)V = f(\epsilon \tau, V, \epsilon), \quad V(0) = \hat{v}(\epsilon) . \]

By differentiating this successively with respect to \( \epsilon \) and setting \( \epsilon = 0 \) in the results, we get

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\( (4a) \quad dV_0/d\tau - A(0, 0)V_0 = f(0, V_0, 0), \quad V_0(0) = \hat{v}(0) \)

\( dV_r/d\tau - [A(0, 0) + f_r(0, V_0, 0)]V_r = \rho_r(\tau), \quad V_r(0) = \hat{v}_r, \)  \( r = 1, 2, \ldots, \)

for the coefficients \( V_r \) of the Taylor expansion of the solution of (3) about \( \epsilon = 0 \). In (4b) \( \hat{v}_r \) is the coefficient of \( \epsilon^r \) in the Taylor expansion about \( \epsilon = 0 \) of \( \hat{v}(\epsilon) \). We observe that for each \( r, R_r \) in (2b) depends only on \( t, v_0, \ldots, v_{r-1} \), and \( \rho_r \) in (4b) depends only on \( \tau, V_0, \ldots, V_{r-1} \).

If problems (2), (4) can be solved successively for the \( v_r, V_r \), we can form the (possibly divergent) expansions

\( (5a) \quad \sum_{r=0}^{\infty} v_r(t)\epsilon^r, \quad (5b) \quad \sum_{r=0}^{\infty} V_r(t/\epsilon)\epsilon^r. \)

From analogy with the ordinary differential equations case (\( A(t, \epsilon) \) bounded operators) the solution of (1) is expected to be represented by (5a) for \( t \) away from zero and by (5b) for \( t \) near zero. To obtain an expansion for the solution of (1) valid uniformly for \( 0 \leq t \leq T \), we employ a matching device. We observe that the expansion

\( (6) \quad \sum_{r=0}^{\infty} v_r(\epsilon t)\epsilon^r \)

formally satisfies the problem (3). Expanding each \( v_r(t) = \sum_{q=0}^{\infty} v_{r,q} t^q \) in its formal Taylor expansion and substituting these into (6) gives

\( (7) \quad \sum_{r=0}^{\infty} U_r(\tau)\epsilon^r, \quad U_r(\tau) = \sum_{q=0}^{r} v_{r-q, q} \tau^q, \quad r = 1, 2, \ldots. \)

It is shown below that expansion (7) is like expansion (5a) for \( t \) near zero and like (5b) for \( t \) away from zero. We thus arrive at the expansion

\( (8) \quad \sum_{r=0}^{\infty} [v_r(t) + V_r(t/\epsilon) - U_r(t/\epsilon)]\epsilon^r. \)

Expansion (8) is considered in three different cases:

(i) Abstract Parabolic Case where for each \( (t, \epsilon) \in [0, T] \times [0, \epsilon_0] \), \(-A(t, \epsilon)\) is the infinitesimal generator of an analytic semigroup of operators in \( E \) (see Kato [1]).

(ii) Abstract Hyperbolic Case where for each \( (t, \epsilon) \), \(-A(t, \epsilon)\) is the infinitesimal generator of a semigroup of class \( C_0 \) [2].

(iii) Parabolic Case where \( A \) is a positive definite elliptic operator in \( E = L^p \).
2. **Assumptions.** We assume:

(II) The problem (2a) has an isolated solution which is infinitely differentiable for $0 \leq t \leq T$ and $f_r(t, v_0(t), 0) = 0$ for $0 \leq t \leq T$ (i.e., $A$ accounts for the linearization of (1) about $v = v_0(t)$).

(III) The problem (4a) has a unique solution $V = V_0(\tau)$ which exists for $0 \leq \tau < \infty$.

Finally, a crucial condition for our work is:

(IV) The resolvent set of $-A(t, e)$ includes the half plane \( \{ \text{Re} Z \geq -\delta \} \) for some $\delta > 0$.

3. **Results.** The proofs of the following theorems will be given elsewhere.

**Theorem 1 (Abstract Parabolic Case).** Let conditions (I)-(IV) and (i) be satisfied. Then for sufficiently small \(|\hat{\delta}(0) - v_0(0)|_E\), there exists a unique solution $v = v(t, e)$ of (1) for each small $e$. Also, the problems (2) and (4) can be solved successively and

$$ v(t, e) = \sum_{r=0}^{\infty} [v_r(t) + V_r(t/e) - U_r(t/e)]e^r $$

where $v_r, V_r, U_r$ are determined from (2), (4) and (7), respectively.

**Remarks.** The notation $g(t, e) \sum_{r=0}^{\infty} \alpha_r(t, e)e^r$ here means that for each $N = 1, 2, \ldots$, the function $S_N$ defined by $e^{N+1}S_N = g(t, e) - \sum_{r=0}^{N} \alpha_r(t, e)e^r$ is bounded in the norm of $E$ uniformly for $0 \leq t \leq T$, $0 < e \leq e_0$.

The restriction on $|\hat{\delta}(0) - v_0(0)|_E$ in Theorem 1 is primarily to ensure that $|V_0(\tau) - v_0(0)|_E \to 0$ as $\tau \to \infty$. Thus, its size depends on the nonlinearity $f$ and the location of the spectrum of $-A(t, e)$.

**Theorem 2. (Abstract Hyperbolic Case).** Let conditions (I)-(IV) and (ii) be satisfied. Also, suppose $|A(t, e) - A(s, e)|_E \leq C|t - s|$ for some constant $C > 0$ (independent of $e$) and all $0 \leq t, s \leq T$. Then the conclusion of Theorem 1 remains valid.

**Remarks.** An important restriction imposed by the continuity condition on $A$ in Theorem 2 is that the operator defined by the difference is a bounded operator. This condition does not appear in the parabolic case because of certain properties of analytic semigroups.

The proof of Theorems 1 and 2 rests on obtaining an estimate for the fundamental solution of problem (1). In particular, we have

**Lemma.** Let either the hypotheses of Theorem 1 or 2 be satisfied. Then
for each \( \varepsilon > 0 \), there is a fundamental solution, \( U(t, s, \varepsilon) \), for the linear part of problem (1) (i.e., (1) with \( f = 0 \)). Moreover there are positive constants \( K, \eta \) such that

\[
| U(t, s, \varepsilon) | \leq K \exp\left[ -\eta(t - s)/\varepsilon \right] \quad \text{for} \ 0 \leq s \leq t \leq T, \ 0 < \varepsilon \leq \varepsilon_0.
\]

The proof of the following theorem rests on obtaining a similar estimate for the fundamental solution when \( E = L^\infty \). Let \( \Omega \) be a bounded domain in Euclidean \( n \)-space \( E^n \) with boundary \( \partial \Omega \) and closure \( \bar{\Omega} \). A point \( x \in E^n \) is given by \( x = (x_1, \ldots, x_n) \), and we use the notation \( D_i = \partial/\partial x_i \). We shall denote by \( \mathfrak{H}(x, t, \varepsilon, D) \) a second order linear differential operator in \( L^\infty \) with real coefficients:

\[
\mathfrak{H}(x, t, \varepsilon, D) = \sum_{i,j=1}^{n} a_{ij}(x, t, \varepsilon) D_i D_j + \sum_{i=1}^{n} a_i(x, t, \varepsilon) D_i + a(x, t, \varepsilon).
\]

The coefficients of \( \mathfrak{H} \) have continuous derivatives of all orders with respect to \( (x, t, \varepsilon) \in \bar{\Omega} \times [0, T] \times [0, \varepsilon_0] \). Also, the matrix \( (a_{ij}) \) is symmetric and positive definite uniformly in \( (x, t, \varepsilon) \) (in particular at \( \varepsilon = 0 \)). Finally, \( \bar{\Omega} \) is of class \( C^2 \).

Consider the initial-boundary value problem

\[
\begin{align*}
eut - \mathfrak{H} u = & \ f(x, t, u, \varepsilon), \quad u = 0 \ on \ \partial \Omega \times [0, T], \\
u(x, 0, \varepsilon) = & \ \hat{u}(x, \varepsilon) \ on \ \bar{\Omega}.
\end{align*}
\]

(9)

Here \( u_t \) denotes \((\partial u/\partial t)\), and we assume \( f, \hat{u} \) have continuous derivatives of all orders. The formal considerations above proceed in the same way for this problem. We will denote by (2*), (4*), etc., those statements reinterpreted for (9). We then have

**Theorem 3 (Parabolic Case).** Let \( \mathfrak{H}, f, \hat{u} \) be as above and let condition (II*)—(IV*) be satisfied. Then for \( |\hat{u}(x, 0) - v_0(x, 0)| \leq L^\infty \) sufficiently small, there is a unique solution of (9) for each small \( \varepsilon \). Moreover, (2*), (4*) can be solved successively and

\[
u(x, t, \varepsilon) = \sum_{r=0}^{\infty} \left[ v_r^*(x, t) + V_r^*(x, t/\varepsilon) - U_r^*(x, t/\varepsilon) \right] \varepsilon^r.
\]

**Remarks.** In (9), \( \mathfrak{H} \) can be replaced by an elliptic operator of order \( 2m \) for any integer \( m > 0 \) with a corresponding change in the boundary conditions (see e.g., Agmon [3]). The estimate in \( L^\infty \) for the fundamental solution of problem (9) is obtained from the maximum principle for parabolic equations.

In [4] Keller formally obtained an expansion for the solution of (9) in the linear case (i.e., \( f \) independent of \( v \)). His expansions involved
the eigenfunctions of $H$. Since the Green's function for the linear part of (9) can be expanded in terms of the eigenfunctions of $H$, the expansion of Theorem 3 can be given in terms of these eigenfunctions. In the linear case the result agrees with that in [4].

The method outlined here is essentially the one developed by Vasil'eva [5] for ordinary differential equations. Her work suggests that these methods can be extended to treat systems of the form

$$u_t = g(t, u, v, \epsilon), \quad \epsilon v_t = A(t, \epsilon)v + f(t, u, v, \epsilon).$$

Such extensions are presently being investigated. Finally, this work has been applied to problems involving the heat equation with nonlinear source in a domain with a slowly moving boundary.

REFERENCES


