CAUCHY PROBLEMS INVOLVING A SMALL PARAMETER
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The purpose of this note is to indicate how certain asymptotic methods developed for ordinary differential equations can be extended and applied to initial-boundary value problems for nonlinear parabolic and hyperbolic equations. This is done by considering the initial-boundary value problem as a Cauchy problem for an ordinary differential equation in an abstract space.

We consider the initial value problem

\[ \varepsilon \frac{dv}{dt} - A(t, \varepsilon)v = f(t, \varepsilon, v, \varepsilon), \quad 0 \leq t \leq T, \quad v(0) = \tilde{v}(\varepsilon) \]

where \( v \) is an element of a Banach space \( E \) and \( \varepsilon > 0 \) is a small parameter. The (possibly unbounded) linear operators \( A \) are assumed to have a common domain of definition \( \mathcal{D} \) independent of \( (t, \varepsilon) \), and the function \( f \) is assumed to have continuous derivatives with respect to \( t, \varepsilon \) and continuous Fréchet derivatives with respect to \( v \). Finally, \( \tilde{v}(\varepsilon) \in \mathcal{D} \) has continuous derivatives with respect to \( \varepsilon \).

We will outline here a method for finding an expansion for the solution of (1) which is valid as \( \varepsilon \to 0 \).

1. **Formal method.** We begin by formally describing the procedure. These steps will be justified by Theorems 1–3. Suppose

(I) the operator \( A(t, \varepsilon) \) has a bounded inverse for each \( (t, \varepsilon) \) and \( A(t, \varepsilon)A^{-1}(0, 0) \) has continuous derivatives with respect to \( (t, \varepsilon) \).

Assuming for the moment that (1) has a solution for \( \varepsilon > 0 \), we differentiate (1) successively with respect to \( \varepsilon \) and set \( \varepsilon = 0 \) in the results. This gives the system of equations

\[
(2a) \quad -A(t, 0)v_0 = f(t, v_0, 0)
\]

\[
(2b) \quad -[A(t, 0) + f_r(t, v_0(t), 0)]v_r = R_r(t), \quad r = 1, 2, \ldots
\]

for the coefficients \( v_r \) of the Taylor expansion of \( v \) about \( \varepsilon = 0 \). Next, we make the change of variables \( t = \varepsilon \tau \) in (1):

\[
(3) \quad \frac{dV}{d\tau} - A(\varepsilon \tau, \varepsilon)V = f(\varepsilon \tau, V, \varepsilon), \quad V(0) = \tilde{v}(\varepsilon).
\]

By differentiating this successively with respect to \( \varepsilon \) and setting \( \varepsilon = 0 \) in the results, we get

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(4a) \[ \frac{dV_0}{d\tau} - A(0, 0)V_0 = f(0, V_0, 0), \quad V_0(0) = \dot{v}(0) \]

\[ \frac{dV_r}{d\tau} - [A(0, 0) + f_r(0, V_0, 0)]V_r = \rho_r(\tau), \quad V_r(0) = \dot{v}_r, \]

(4b) \[ r = 1, 2, \ldots, \]

for the coefficients \( V_r \) of the Taylor expansion of the solution of (3) about \( \epsilon = 0 \). In (4b) \( \dot{v}_r \) is the coefficient of \( \epsilon^r \) in the Taylor expansion about \( \epsilon = 0 \) of \( \dot{v}(\epsilon) \). We observe that for each \( r \), \( R_r \) in (2b) depends only on \( t, v_0, \ldots, v_{r-1} \), and \( \rho_r \) in (4b) depends only on \( \tau, V_0, \ldots, V_{r-1} \).

If problems (2), (4) can be solved successively for the \( v_r \), \( V_r \), we can form the (possibly divergent) expansions

\[ \sum_{r=0}^{\infty} v_r(t)\epsilon^r, \quad \sum_{r=0}^{\infty} V_r(t/\epsilon)\epsilon^r. \]

From analogy with the ordinary differential equations case \( (A(t, \epsilon) \) bounded operators) the solution of (1) is expected to be represented by (5a) for \( t \) away from zero and by (5b) for \( t \) near zero. To obtain an expansion for the solution of (1) valid uniformly for \( 0 \leq t \leq T \), we employ a matching device. We observe that the expansion

\[ \sum_{r=0}^{\infty} v_r(\epsilon \tau)\epsilon^r \]

formally satisfies the problem (3). Expanding each \( v_r(t) = \sum_{q=0}^{\infty} v_{r,q} t^q \) in its formal Taylor expansion and substituting these into (6) gives

\[ \sum_{r=0}^{\infty} U_r(\tau)\epsilon^r, \quad U_r(\tau) = \sum_{q=0}^{r} v_{r-q,q} \tau^q, \quad r = 1, 2, \ldots. \]

It is shown below that expansion (7) is like expansion (5a) for \( t \) near zero and like (5b) for \( t \) away from zero. We thus arrive at the expansion

\[ \sum_{r=0}^{\infty} [v_r(t) + V_r(t/\epsilon) - U_r(t/\epsilon)]\epsilon^r. \]

Expansion (8) is considered in three different cases:

(i) Abstract Parabolic Case where for each \( (t, \epsilon) \in [0, T] \times [0, \epsilon_0] \), \(-A(t, \epsilon) \) is the infinitesimal generator of an analytic semigroup of operators in \( E \) (see Kato [1]).

(ii) Abstract Hyperbolic Case where for each \( (t, \epsilon) \), \(-A(t, \epsilon) \) is the infinitesimal generator of a semigroup of class \( C_0 \) [2].

(iii) Parabolic Case where \( A \) is a positive definite elliptic operator in \( E = L^p \).
2. **Assumptions.** We assume:

(I) The problem (2a) has an isolated solution which is infinitely differentiable for \(0 \leq t \leq T\) and \(f_0(t, v_0(t), 0) = 0\) for \(0 \leq t \leq T\) (i.e., \(A\) accounts for the linearization of (1) about \(v = v_0(t)\)).

(III) The problem (4a) has a unique solution \(V = V_0(\tau)\) which exists for \(0 \leq \tau < \infty\).

Finally, a crucial condition for our work is:

(IV) The resolvent set of \(-A(t, \epsilon)\) includes the half plane \(\{\text{Re}\, Z \geq -\delta\}\) for some \(\delta > 0\).

3. **Results.** The proofs of the following theorems will be given elsewhere.

**Theorem 1 (Abstract Parabolic Case).** Let conditions (I)–(IV) and (i) be satisfied. Then for sufficiently small \(|\hat{\phi}(0) - v_0(0)|_E,\) there exists a unique solution \(v = v(t, \epsilon)\) of (1) for each small \(\epsilon\). Also, the problems (2) and (4) can be solved successively and

\[
v(t, \epsilon) = \sum_{n=0}^{\infty} [v_n(t) + V_n(t/\epsilon) - U_n(t/\epsilon)] \epsilon^n
\]

where \(v_n, V_n, U_n\) are determined from (2), (4) and (7), respectively.

**Remarks.** The notation \(g(t, \epsilon) = \sum_{\tau=0}^{\infty} a_\tau(t, \epsilon) \epsilon^\tau\) here means that for each \(N = 1, 2, \ldots\), the function \(S_N\) defined by \(e^{N+1}S_N = g(t, \epsilon) - \sum_{\tau=0}^{N} a_\tau(t, \epsilon) \epsilon^\tau\) is bounded in the norm of \(E\) uniformly for \(0 \leq t \leq T, 0 < \epsilon \leq \epsilon_0\).

The restriction on \(|\hat{\phi}(0) - v_0(0)|_E\) in Theorem 1 is primarily to ensure that \(|V_0(\tau) - v_0(0)|_E \rightarrow 0\) as \(\tau \rightarrow \infty\). Thus, its size depends on the nonlinearity \(f\) and the location of the spectrum of \(-A(t, \epsilon)\).

**Theorem 2. (Abstract Hyperbolic Case).** Let conditions (I)–(IV) and (ii) be satisfied. Also, suppose \(|A(t, \epsilon) - A(s, \epsilon)|_E \leq C|t - s|\) for some constant \(C > 0\) (independent of \(\epsilon\)) and all \(0 \leq t, s \leq T\). Then the conclusion of Theorem 1 remains valid.

**Remarks.** An important restriction imposed by the continuity condition on \(A\) in Theorem 2 is that the operator defined by the difference is a bounded operator. This condition does not appear in the parabolic case because of certain properties of analytic semigroups.

The proof of Theorems 1 and 2 rests on obtaining an estimate for the fundamental solution of problem (1). In particular, we have

**Lemma.** Let either the hypotheses of Theorem 1 or 2 be satisfied. Then
for each \( \varepsilon > 0 \), there is a fundamental solution, \( U(t, s, \varepsilon) \), for the linear part of problem (1) (i.e., (1) with \( f = 0 \)). Moreover there are positive constants \( K, \eta \) such that

\[
|U(t, s, \varepsilon)| \leq K \exp[-\eta(t - s)/\varepsilon] \quad \text{for} \ 0 \leq s \leq t \leq T, \quad 0 < \varepsilon \leq \varepsilon_0.
\]

The proof of the following theorem rests on obtaining a similar estimate for the fundamental solution when \( E = L^\infty \). Let \( \Omega \) be a bounded domain in Euclidean \( n \)-space \( E^n \) with boundary \( \partial \Omega \) and closure \( \bar{\Omega} \). A point \( x \in E^n \) is given by \( x = (x_1, \ldots, x_n) \), and we use the notation \( D_i = \partial/\partial x_i \). We shall denote by \( \mathcal{A}(x, t, \varepsilon, D) \) a second order linear differential operator in \( L^\infty \) with real coefficients:

\[
\mathcal{A}(x, t, \varepsilon, D) = \sum_{i,j=1}^n a_{ij}(x, t, \varepsilon) D_i D_j + \sum_{i=1}^n a_i(x, t, \varepsilon) D_i + a(x, t, \varepsilon).
\]

The coefficients of \( \mathcal{A} \) have continuous derivatives of all orders with respect to \( (x, t, \varepsilon) \in \bar{\Omega} \times [0, T] \times [0, \varepsilon_0] \). Also, the matrix \( (a_{ij}) \) is symmetric and positive definite uniformly in \( (x, t, \varepsilon) \) (in particular at \( \varepsilon = 0 \)). Finally, \( \mathcal{A} \) is of class \( C^2 \).

Consider the initial-boundary value problem

\[
\begin{align*}
\mathcal{A}(x, t, \varepsilon) u_t - \Delta u &= f(x, t, u, \varepsilon), \quad u = 0 \text{ on } \partial \Omega \times [0, T], \\
u(x, 0, \varepsilon) &= \hat{u}(x, \varepsilon) \text{ on } \bar{\Omega}.
\end{align*}
\]

(9)

Here \( u_t \) denotes \( \partial u / \partial t \), and we assume \( f, \hat{u} \) have continuous derivatives of all orders. The formal considerations above proceed in the same way for this problem. We will denote by (2*), (4*), etc., those statements reinterpreted for (9). We then have

**Theorem 3 (Parabolic Case).** Let \( \mathcal{A}, f, \hat{u} \) be as above and let condition (II*)—(IV*) be satisfied. Then for \( |\hat{u}(x, 0) - v_0(x, 0)|_{L^\infty} \) sufficiently small, there is a unique solution of (9) for each small \( \varepsilon \). Moreover, (2*), (4*) can be solved successively and

\[
u(x, t, \varepsilon) = \sum_{r=0}^\infty \left[ v_r^*(x, t) + V_r^*(x, t/\varepsilon) - U_r^*(x, t/\varepsilon) \right] \varepsilon^r.
\]

**Remarks.** In (9), \( \mathcal{A} \) can be replaced by an elliptic operator of order \( 2m \) for any integer \( m > 0 \) with a corresponding change in the boundary conditions (see e.g., Agmon [3]). The estimate in \( L^\infty \) for the fundamental solution of problem (9) is obtained from the maximum principle for parabolic equations.

In [4] Keller formally obtained an expansion for the solution of (9) in the linear case (i.e., \( f \) independent of \( u \)). His expansions involved
the eigenfunctions of \( \mathcal{A} \). Since the Green's function for the linear part of (9) can be expanded in terms of the eigenfunctions of \( \mathcal{A} \), the expansion of Theorem 3 can be given in terms of these eigenfunctions. In the linear case the result agrees with that in [4].

The method outlined here is essentially the one developed by Vasil'eva [5] for ordinary differential equations. Her work suggests that these methods can be extended to treat systems of the form

\[
\begin{align*}
\frac{u_t}{t} &= g(t, u, v, \epsilon), \\
\epsilon v_t &= A(t, \epsilon)v + f(t, u, v, \epsilon).
\end{align*}
\]

Such extensions are presently being investigated. Finally, this work has been applied to problems involving the heat equation with nonlinear source in a domain with a slowly moving boundary.

REFERENCES


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