ON THE SPAN OF A RIEMANN SURFACE

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In 1952 Virtanen [7] gave an example of a Riemann surface which carries nonconstant AD-functions, yet there is a point on the surface where every such function has a critical value, i.e. a vanishing derivative. It had already been pointed out in [1] that this phenomenon cannot occur for planar surfaces. It has remained an open question whether this situation prevails for harmonic functions. This unsolved problem was restated in the monograph *Capacity functions* (Sario-Oikawa [5]). Problem 4 in the list of open questions at the end of that book asks, more generally, whether the vanishing of the span S_m at one point on a Riemann surface implies that it vanishes at all points on the surface.

In this paper we describe a Riemann surface R with the property that for any preassigned positive integer m there is a point on R where every HD-function has a critical point of order m. Yet R carries nonconstant HD-functions. Thus Problem 4 of [5] is solved for the H-span. Nevertheless, there are still unresolved questions in this area when we restrict ourselves to surfaces of finite genus or to KD-functions.

Consider the surface of Tôki [6] (this surface is also constructed in Ahlfors-Sario [2]; see No. 25 of §8, Chapter IV). This surface is realized as a unit disk with infinitely many radial slits identified in a prescribed manner. It is of class $O_{HD}-O_G$, and hence the Royden harmonic boundary [4] consists of exactly one point.

Remove a disk $\{|z|<\epsilon\}$ from Tôki's surface and then form the double across $\{|z|=\epsilon\}$. The resulting surface is realized as a radial slit annulus $\{\epsilon^2<|z|<1\}$ with certain identifications among the edges of the slits. For a preassigned positive integer m, insert an additional $m+1$ radial slits symmetric to the origin and identify their edges cyclically. The resulting surface R has a branch point of order m, and the Royden harmonic boundary of R consists of two points. Therefore the space of HD-functions on R is two-dimensional.

The function $\log r$ can be regarded as an HD-function on R, and in that sense it has critical point of order m at the branch point of order m. The same is true for any HD-function u on R since

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$$u = c_1 \log r + c_2.$$

We see that $R \notin O_{HD}$, yet there is a point on R where the m-span S_m vanishes.

REFERENCES

1. L. V. Ahlfors and A. Beurling, *Conformal invariants and function-theoretic null-sets*, Acta Math. **83** (1950), 101–129. MR **12**, 171.
2. L. V. Ahlfors and L. Sario, *Riemann surfaces*, Princeton Math. Series, no. 26, Princeton Univ. Press, Princeton, N. J., 1960. MR **22** #5729.
3. L. Sario, *Über Riemannsche Flächen mit hebbarem Rand*, Ann. Acad. Sci. Fenn. Ser. A. I. Math.-Phys. No. 50 (1948). MR **10**, 365.
4. H. L. Royden, *On the ideal boundary of a Riemann surface. Contributions to the theory of Riemann surfaces*, Ann. of Math. Studies, no. 30, Princeton Univ. Press, Princeton, N. J., 1953. MR **15**, 25.
5. L. Sario and K. Oikawa, *Capacity functions*, Springer-Verlag, Berlin and New York, 1969.
6. Y. Tôki, *On the classification of open Riemann surfaces*, Osaka Math. J. **4** (1952), 191–201. MR **14**, 864.
7. K. I. Virtanen, *Über Extramalfunktionen auf offenen Riemannschen Flächen*, Ann. Acad. Sci. Fenn. Ser. A. I. Math.-Phys. No. 141 (1952). MR **14**, 743.

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