BAER SUBPLANES AND BLOCKING SETS

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A *blocking set* S in a projective plane π is a subset of the points of π such that every line of π contains at least one point of S and at least one point which is not in S. Denoting the number of points in S by $|S|$ our main result, obtained by purely combinatorial means, is the following: If π is finite of square order, say m^2 then $|S| \geqq m^2+m+1$ and if $|S| = m^2+m+1$ then the points of S are the points of a subplane of π of order m (a Baer subplane). In this connection we first of all prove the following

THEOREM. *Baer subplanes form blocking sets.*

PROOF. Suppose π is a plane of order m^2 which contains a subplane S of order m. Since any line of π contains at most $m+1$ points of S we have that every line of π contains at least one point which is not in S. Let l be any line of π and P be any point of l which is not in S. Then there is at most one line of S through P, S being a subplane. Also since any two points of π are connected by a unique line, the m^2+m+1 points of S are contained in the m^2+1 lines of π through P. If l contained no point of S, the lines of π through P would account for at most $(m+1)+(m^2-1)\cdot 1 = m^2+m$ points of S. Thus l must contain at least one point of S establishing our theorem.

We now proceed to the main result. π denotes a plane of order n and S is a blocking set in π. $S-l$ denotes all those points P such that P is contained in S but not in l, and $|S-l|$ means the number of such points P; similarly for $l-S$, $|l-S|$.

LEMMA 1. *No line of π contains more than $|S| - n$ points of S.*

PROOF. Let l be any line of π and suppose l contains exactly t points of S. Since S is a blocking set there is at least one point R in $l-S$. There are n lines of π through R besides l, each containing at least one point of S. Thus always $|S| \geqq t+n$.

LEMMA 2. *Let a objects be packed into b boxes such that each box contains at least one object, with $b \leqq a < 2b$. Define a function f on the objects X as follows:*

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$f(X)=1$ if the box containing X contains other objects,
$f(X)=0$ otherwise.

Then for any such packing P, we have

$$A(P) = \sum f(X) \leqq 2(a-b),$$

the summation being over all objects X.

PROOF. It can be seen that if some box contains more than two objects, for some packing P, there is a packing P' such that $A(P') > A(P)$. Hence $A(P)$ attains its maximum value when each box contains no more than two objects, and, in this case, $A(P)=2(a-b)$.

From now on we assume that n is a square, $n=m^2$, say.

LEMMA 3. *Suppose $|S|=m^2+m+1$. Let some line l of π contain exactly k points of S. Let B denote those lines of π passing through points of $l-S$ and containing at least two points of $S-l$. Let I denote the set of incidences of points of $S-l$ with lines of B. Then $|I| \leqq 2(m+1-k)$ m^2+1-k.*

PROOF. For each point P in $l-S$ the $m^2+m+1-k$ points of $S-l$ are packed into m^2 lines through P. Hence, by Lemma 2 these lines through P yield at most $2[(m^2+m+1-k)-m^2]$ incidences in I. Thus, since $|l-S|=m^2+1-k$, we have

$$|I| \leqq 2(m^2+1-k)(m+1-k).$$

LEMMA 4. *If $|S|=m^2+m+1$, some line of π contains precisely $m+1$ points of S.*

PROOF. Let some line l of π contain precisely k points of S where k is the maximum number of points of S on any line of π. Clearly $k \geqq 2$ and, by Lemma 1, $k \leqq m+1$. Let B, I be as in Lemma 3, and P any point of $S-l$. There remain m^2+m-k points of $S-l$ and the k lines of π which connect P to points of $S \cap l$ account for at most $k(k-2)$ of them. Thus there are at least $m^2+m-k-k(k-2)$ points of $S-l$ different from P and also incident with lines of B through P. If there are b lines of B through P we must have $b(k-1) \geqq [m^2+m-k -k(k-2)]$. Thus the lines of B through P yield at least b incidences in I, where $b \geqq (m+1-k)(m+k)(k-1)^{-1}$. Summing over all the points of $S-l$ such as P we obtain $|I| \geqq (m^2+m+1-k)b$. Thus, from Lemma 3, we must have

$$2(m^2+1-k)(m+1-k) \geqq (m^2+m+1-k)b.$$

If we assume $k<m+1$ we have $2(k-1)(m^2+1-k) \geqq (m^2+m+1-k)$

$(m+k)$. Now, $k \leqq m \Rightarrow 2(k-1) < 2k \leqq (m+k)$ and $m^2+1-k < m^2+m+1-k$, that is, the supposition $k < m+1$ is contradictory. Thus, from Lemma 1, $k = m+1$, and some line of π contains precisely $m+1$ points of S.

THEOREM 1. *If* $|S| = m^2+m+1$, *then the points of S are the points of a Baer subplane of* π.

PROOF. By Lemma 4 some line l of π contains precisely $m+1$ points of S. Since S is a blocking set, we have that if U and V are any two distinct points of $S-l$ the line UV of π must meet l in a point of $S \cap l$. Thus for any point P of $S-l$ the $(m+1)$ lines of π connecting P to the $m+1$ points of $S \cap l$ account for all the m^2 points of $S-l$, and, using Lemma 1, each such line contains precisely $m+1$ points of S. Hence if we define a structure π' such that the points of π' are the points of S, the lines of π' are those lines of π containing at least two points of S, and incidence in π' is given by incidence in π, it can be seen that π' is a subplane of π, and π' has order m.

THEOREM 2. $|S| \geqq m^2+m+1$.

Suppose $|S| = m^2+m+1-t$, $t > 0$. By Lemma 1 no line of π contains more than $m+1-t$ points of S. Let L be any set of t points of π none of which is in S and such that the points of S' do not form the points of a Baer subplane of π where $S' = S \cup L$. Then S' is a blocking set since no line of π contains more than $(m+1-t)+t$ points of S'. Thus we would have a blocking set S' with $(S') = m^2+m+1$; by the condition on L this contradicts Theorem 1.

REMARK. The author has since proved that $|S| \geqq n+n^{1/2}+1$ for π of order n, n arbitrary. This result and some corollaries will be discussed elsewhere.

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