BOUNDING IMMERSIONS OF CODIMENSION 1 IN THE EUCLIDEAN SPACE

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Let $M$ be an $(n+1)$-dimensional differentiable manifold without boundary (compact or not) and $f: V \rightarrow M$ an immersion of the compact $n$-dimensional manifold without boundary $V$. We say that $f$ is a bounding immersion if there is a manifold $W^{n+1}$ with boundary $dW = V$, and an immersion $g: W \rightarrow M$ such that $f = g|V$. If $M$ and $V$ are oriented, then $V$ must be the oriented boundary of the oriented manifold $W$, and $g$ an oriented immersion of codimension 0.

Using the classification of immersions (Smale [7], Hirsch [2]) and the work of Kervaire-Milnor [3], [4], we compute in this note the regular homotopy classes of all bounding immersions of the sphere $S^n$ into the euclidean space $\mathbb{R}^{n+1}$ and into the sphere $S^{n+1}$.

1. Statement of the results. From [2] we know that the derivation $f \mapsto T(f)$ defines a weak homotopy equivalence between the space $\text{Imm}(V, M)$ of the immersions of $V$ into $M$ and the space of the fibre-maps of the tangent bundle $T(V)$ into the tangent bundle $T(M)$ which are injective in each fibre. If $V = S^n$ and $M = \mathbb{R}^{n+1}$, the set of connected components of this last space is an homogeneous space under the group $\pi_n(SO(n+1))$. By a convenient identification, we obtain a bijection $\gamma: \pi_0(\text{Imm}(S^n, \mathbb{R}^{n+1})) \rightarrow \pi_n(SO(n+1))$ such that the class of the ordinary imbedding be $0 \in \pi_n(SO(n+1))$. Furthermore the map $\gamma$ is additive with respect to the connected sum of immersions [5].

Similarly, using the fact that the fibration $SO(n+2) \rightarrow S^{n+1} = SO(n+2)/SO(n+1)$ is the principal fibration with group $SO(n+1)$ tangent to $S^{n+1}$, it is easy to obtain a bijection $\beta: \pi_0(\text{Imm}(S^n, S^{n+1})) \rightarrow \pi_n(SO(n+2))$ additive with respect to the connected sum. If $i: \mathbb{R}^{n+1} \rightarrow S^{n+1}$ is the stereographic projection with the south pole $(x_1 = -1)$ as center, we have a commutative diagram

$$
\begin{array}{ccc}
\pi_0(\text{Imm}(S^n, \mathbb{R}^{n+1})) & \gamma & \pi_n(SO(n+1)) \\
\downarrow i_* & & \downarrow s \\
\pi_0(\text{Imm}(S^n, S^{n+1})) & \beta & \pi_n(SO(n+2))
\end{array}
$$

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where the stabilization homomorphism $s$ is induced by the inclusion of $SO(n+1)$ as the subgroup of $SO(n+2)$ acting on the $n+1$ last coordinates. From now on, we shall denote by $\bar{f}$ the class $\gamma(f)$.

If we denote by $J_n: \pi_n(SO(n+2)) \to \pi_n (= \pi_{2n+2}(S^{n+2}))$ the stable Hopf-Whitehead homomorphism, we can state the result:

**Theorem 1.** For each $n \geq 1$, the set of the classes of the bounding immersions of $S^n$ in $S^{n+1}$ is the kernel of $J_n$.

**Theorem 2.** For each $n \geq 2$, the set of the classes of the bounding immersions of $S^n$ in $R^{n+1}$ is the kernel of $J_n \circ s$.

In order to prove those results, we admit some lemmas whose proof will appear elsewhere.

2. **First step of the proof.** Let $A_{n+1}$ be the cobordism group of stably parallelized manifolds $W^{n+1}$ with boundary $S^n$. If $W$ is the manifold without boundary obtained from $W$ by gluing a disk $D^{n+1}$ along the boundary $S^n = dW$, we denote by $a(W, T) \in \pi_n(SO(n+2))$ the obstruction to extend the $s$-parallelization $T$ of $W$ to $W'$: hence we have an homomorphism $a: A_{n+1} \to \pi_n(SO(n+1))$. Similarly, let $B_{n+1}$ be the monoid of isomorphism classes of such manifolds $W$ with a true parallelization. It follows from [2] or [6] that, if $t$ is a parallelization of $W$, there is an immersion $g: W \to R^{n+1}$, unique up to regular homotopy, such that the trivialization $T(g)$ of $T(W)$ be homotopic to $t$. If we consider the class of the restriction $f$ of $g$ to $dW = S^n$, we define an homomorphism $b: B_{n+1} \to \pi_n(SO(n+1))$. Furthermore we have a natural homomorphism $S: B_{n+1} \to A_{n+1}$.

**Lemma 1.** The following diagram

$$
\begin{array}{ccc}
B_{n+1} & \rightarrow & \pi_n(SO(n+1)) \\
S \downarrow & & \downarrow \\
A_{n+1} & \rightarrow & \pi_n(SO(n+2))
\end{array}
$$

is commutative.

Thus, the set of classes of bounding immersions in $Imm(S^n, R^{n+1})$, which is the image of $b$, is a monoid included in $\ker(J_n \circ s)$ because of the exactness of the sequence

$$
A_{n+1} \rightarrow \pi_n(SO(n+2)) \rightarrow \pi_n
$$

(see [4]); and $b(B_{n+1})$ intersects each fibre $s^{-1}(x), x \in \ker(J_n)$, since the map $S$ is surjective. To prove Theorem 2, it suffices to prove that $b(B_{n+1})$ contains $\ker(s)$ (if $n \geq 2$).
3. Second step. Let \( u \in \pi_n(\text{SO}(n+1)) \) be the boundary of the generator \( i_{n+1} \in \pi_{n+1}(S^{n+1}) \) in the homotopy exact sequence

\[
\pi_{n+1}(S^{n+1}) \xrightarrow{d} \pi_n(\text{SO}(n+1)) \xrightarrow{s} \pi_n(\text{SO}(n+2)) \to 0
\]

of the fibration \( S^{n+1} = \text{SO}(n+2)/\text{SO}(n+1) \). The cyclic group \( \text{Ker}(s) \) is generated by \( u \). From the following lemma and the fact that there are parallelizable closed manifolds in all dimensions, it results that \( u \) is the class of a bounding immersion:

**Lemma 2.** If \( W'' \) is a closed parallelizable closed manifold, and \( t \) is the restriction to \( W = W' - D^{n+1} \) of a parallelization of \( W' \), then \( b(W, t) = u \in \pi_n(\text{SO}(n+1)) \).

Now, we can prove Theorem 1. First, we remark that any immersion \( F: S^n \to S^{n+1} \) is regular homotopic to an immersion \( i \circ f \), where \( f \in \text{Imm}(S^n, R^{n+1}) \) and that \( i \circ f \) and \( i \circ f' \) have the same class in \( \text{Imm}(S^n, S^{n+1}) \) if and only if there is some \( q \in \mathbb{Z} \) such that \( f' = f + qu \). Then we remark that, if \( F \) is a bounding immersion in \( S^{n+1} \), it is regular homotopic to an immersion \( F' \) bounded by \( G' : W \to S^{n+1} \) whose image \( G'(W) \) avoids the south pole. Therefore:

**Lemma 3.** Let \( f \in \text{Imm}(S^n, R^{n+1}) \); the following assertions are equivalent:

(i) \( J_n \circ s(f) = 0 \).

(ii) There is a bounding immersion regular homotopic (in \( S^{n+1} \)) to \( i \circ f \).

(iii) There is a bounding immersion \( f' \in \text{Imm}(S^n, R^{n+1}) \) such that \( f' = f + qu \) for some \( q \in \mathbb{Z} \).

Theorem 1 is a quite evident consequence of Lemma 3.

4. Last step. If \( n \) is even, Theorem 2 is already proved, because \( \text{Ker}(s) \) contains at most the two elements \( 0 \) and \( u \) which are both bounding. If \( n = 2 \) or \( 6 \), then \( \pi_n(\text{SO}(n+1)) = 0 \) and the only class is trivially the class of a bounding immersion. If \( n \neq 2, 6 \), then \( J_n \) is injective \([1]\) and the two distinct classes \( 0 \) and \( u \) are the only bounding classes.

If \( n \) is odd, the kernel of \( s \) is infinite cyclic, generated by \( u \) and it suffices to prove that \( -u \) is the class of a bounding immersion, since \( b(B_{n+1}) \) is a monoid.

If \( f \in \text{Imm}(S^n, R^{n+1}) \), let \( d(f) \in \mathbb{Z} \) be the normal degree (curvatura integra) of the immersion \( f \) (see \([5]\)). It is proved in \([5]\) that \( d(f' - f) = d(f) + d(f') - 1 \). Now, the Hopf theorem of curvatura integra states that \( d(f) = \chi(W) \) if \( f \) is the restriction to the boundary of an immersion.
It is clear that \( d(0) = 1 \), and it follows from Lemma 2 that 
\[ d(u) = -1. \]
Thus, the elements \( qu (q \in \mathbb{Z}) \) of \( \text{Ker}(s) \) are determined by their (odd) degree 
\[ d(q \cdot u) = 1 - 2q. \]

If \( n = 1 \), there is no 2-manifold, with boundary \( S^1 \), whose Euler number is more than 1, so that:

**Theorem 2'.** In \( \pi_0(\text{Imm}(S^1, \mathbb{R}^3)) \cong \pi_1(\text{SO}(2)) \), the classes of bounding immersions are the classes of odd degree \( 1 - 2q, q \geq 0 \).

For \( n \) odd \( \neq 1 \), the manifold \( W' = S^2 \times S^{n-1} \) is \( s \)-parallelizable; there is a parallelization \( t \) of the manifold \( W = W' - D^{n+1} \) which stably extend to \( W' \). It follows from Lemma 1 that \( b(W, t) \in \text{Ker}(s) \). Now, the Euler number of \( W \) is 3 so that \( b(W, t) = -u \). Thus, \( -u \) is the class of a bounding immersion and Theorem 2 is proved.

5. Application.

**Theorem 3.** Let \( V^n \) be an \( s \)-parallelizable compact manifold without boundary, and \( f: V \to \mathbb{R}^{n+1} \) an immersion. Suppose \( n \geq 2 \). If \( i \circ f: V \to S^{n+1} \) is a bounding immersion, then \( f \) is regular homotopic (in \( \mathbb{R}^{n+1} \)) to an immersion \( f' \) which is bounding (in \( \mathbb{R}^{n+1} \)).

If the manifold \( V \) is the \( n \)-sphere, this theorem is an immediate corollary of Theorems 1 and 2. In the general case, we deform the immersion \( G: W \to S^{n+1} \) which bounds \( F = i \circ f \) in an immersion \( G' \) whose image \( G'(W) \) avoid the south pole, so that \( G' = i \circ g' \). The immersion \( g' \) bounds \( f' \) such that \( i \circ f' = F' = G'|V \). But \( f \) and \( f' \) have not the same class (in \( \mathbb{R}^{n+1} \)) because, during the regular homotopy, the class of \( f \) has been changed by each crossing of the south pole.

Let \( F_t: V \to S^{n+1} (t \in [0, 1]) \) be a regular homotopy with only one crossing of the south pole through \( F_t(V) \), then \( F_1 \) is regular homotopic to the connected sum \( f_0 + h \) of \( f_0 \) with an immersion \( h: S^n \to \mathbb{R}^{n+1} \) with class \( h \in \text{Ker}(s) \) (in fact, \( h = \pm u \), depending on the direction of the crossing).

Thus, \( f \) is regular homotopic to an immersion \( f'' \) which is the connected sum of \( f' \) with some immersions \( h_i \) such that \( h_i \in \text{Ker}(s) \). We can replace the \( h_i \) by bounding immersions \( k_i \) of the same class (Theorem 2), and, now, \( f'' \) is the connected sum of the bounding immersions \( f' \) and \( k_i \); so \( f'' \) is a bounding immersion.

**Bibliography**


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