A CLASSIFICATION OF MODULES OVER COMPLETE DISCRETE VALUATION RINGS

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1. **Introduction.** The purpose of this paper is to announce the completion of a classification (up to isomorphism) of all modules which are direct sums of countably generated modules over complete discrete valuation rings. The detailed proofs will appear elsewhere. Throughout this paper, let R denote a fixed but arbitrary complete discrete valuation ring and p a fixed but arbitrary prime element of R. For the sake of convenience, a cardinal is viewed as the first ordinal having that cardinality. Let (c, R, k) be the class of all countably generated reduced R-modules of (torsion-free) rank $\leqq k$ and $D(c, R, k)$ that of all direct sums of members of (c, R, k). Clearly

$$\begin{array}{ccccccc} (c, R, 0) & \subset & (c, R, 1) & \subset & \cdots & \subset & (c, R, \omega) \\ \cap & & \cap & & & & \cap \\ D(c, R, 0) & \subset & D(c, R, 1) & \subset & \cdots & \subset & D(c, R, \omega). \end{array}$$

Notice that a p-primary abelian group is a member of $(c, R, 0)$, particularly if R is a ring of p-adic integers. A classification (of all members) of (c, R, k) was done by Ulm (1933) when $k=0$ [**8**], by Kaplansky and Mackey (1951) when $k=1$ [**4**], by Rotman and Yen (1961) when $k<\omega$ [**7**], and that of $D(c, R, k)$ was done by Kolettis (1960) when $k=0$ [**5**]. First, we complete a classification of (c, R, ω) and then, utilizing this, we finish that of $D(c, R, \omega)$.

2. **Invariants.** We need two kinds of invariants, namely, the Ulm invariants and the basis types. Since the celebrated Ulm invariants are well known, a brief explanation of the basis types only is in order [2], [4], [7]. Let $R^k = \oplus \{R : i < k\}$ for each k. Define $f(R)$ to be the class of all sordinal (ordinal or ∞) valued functions on R^k for all cardinals k, and $m(Q)$ that of all square row-finite matrices over Q, the quotient field of R. Suppose that $f, g \in f(R)$. Define $f \sim g$ to mean

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both that Dom $f=$ Dom $g=R^k$ for some cardinal k and that there is a matrix γ and a diagonal matrix δ, both $k\times k$ invertible integral (that is, all entries are elements of R) in $m(Q)$, such that $f(\alpha\gamma)=g(\alpha\delta)$ for all $\alpha\in R^k$. It is routine to show that $\sim$ is an equivalence relation on $f(R)$.

Let M be an R-module of rank k. Then, every basis $\eta=\{y_i:i<k\}$ defines a function g of $f(R)$ by

$$g(\alpha) = h_p(\alpha\eta) = h_p\left(\sum\{a_iy_i: i<k\}\right)$$

for all $\alpha=\{a_i:i<k\}\in R^k$. Notice that $g=\infty$ if $k=0$ since a sum without term is 0. It is routine to show that $g\sim g'$ if g' is defined by another basis of M. Thus, M determines uniquely a class of $f(R)/\sim$, which we call the *basis type* of M. It is easy to show the following lemma.

LEMMA 1. *Two reduced R-modules M and M' have the same basis type if and only if they contain basic free submodules F and F', respectively, with a height-preserving isomorphism from F onto F'.*

3. **A classification of (c, R, ω).**

THEOREM 1. *Let M and M' be countably generated reduced R-modules. Then, $M\simeq M'$ if and only if they have the same Ulm invariant and the same basis type.*

Only the "if" part needs a proof. Let $\alpha=\{a_i:i<k\}$. Define $\alpha(r)=\{a_i:i<r\}$ for each number r. Let k be the same rank of M and M'. Then, by Lemma 1, there are ordered bases $\eta=\{y_i:i<k\}$ and $\eta'=\{y_i':i<k\}$ of M and M', respectively, with a height-preserving isomorphism ρ such that $\rho(\alpha\eta)=\alpha\eta'$ for all $\alpha\in R^k$. We may assume that there are countable subsets $\xi=\{x_i:i<\omega\}$ and $\xi'=\{x_i':i<\omega\}$ of M and M', respectively, such that

$$M=[\xi\cup\eta] \quad\text{and}\quad M'=[\xi'\cup\eta'] \qquad\text{with}$$

$$px_i\in[\xi(i)\cup\eta(i)] \quad\text{and}\quad px_i'\in[\xi'(i)\cup\eta'(i)] \qquad\text{for each } i<\omega.$$

The main idea of the proof is to construct a sequence of height-preserving isomorphisms $\{\phi_i:i<\omega\}$ in such a way that the following conditions are satisfied.

(a) $\phi_i:A_i\rightarrowtail\!\!\twoheadrightarrow A_i'$ where

$$A_i=[\xi(i)\cup\eta(i)\cup\phi_i^{-1}(\xi'(i)\cup\eta'(i))],$$
$$A_i'=[\xi'(i)\cup\eta'(i)\cup\phi_i(\xi(i)\cup\eta(i))].$$

(b) $\phi_0\leqq\cdots\leqq\phi_i\leqq\phi_{i+1}\leqq\cdots$.

(c) There exists a nonnegative integer $n(i)$ such that $p^{n(i)}A_i$

$\subseteq [\eta(i)]$ and $p^{n(i)}A'_i = [\eta'(i)]$ and $\phi_i = \rho$ as height-preserving isomorphism from $p^{n(i)}A_i$ onto $p^{n(i)}A'_i$.

The supremum of $\{\phi_i : i < \omega\}$ gives the required isomorphism from M onto M'. For more detailed proof, see [1] or [2].

4. A classification of $D(c, R, \omega)$.

THEOREM 2. *Let M and M' be direct sums of countably generated reduced R-modules. Then, $M \simeq M'$ if and only if they have the same Ulm invariant and the same basis type.*

Again, only the "if" part needs a proof. We may write as

$$M = \oplus \{M_i : i \in I\} \quad \text{and} \quad M' = \oplus \{M'_i : i \in I\}$$

where all $M_i, M'_i \in (c, R, \omega)$ and I is a cardinal. For notational convenience, define $M(T) = \oplus \{M_i : i \in T\}$, $T \subseteq I$. The main idea of the proof is to show that there is a partition of I into countable subsets $\{I_j : j < I\}$ such that, for each $j < I$, $M(I_j)$ and $M'(I_j)$ have the same Ulm invariant and the same basis type. Then by Theorem 1, they are isomorphic and, consequently, $M \simeq M'$. In fact, by the Kolettis theorem [3], [5], [6], we may assume that M_i and M'_i have already the same Ulm invariant for each i. The following lemmas indicate the route of the proof.

LEMMA 2. *Let $N = A \oplus B \oplus C$ be a reduced R-module such that the following conditions are satisfied.*

(a) *There are in N disjoint subsets η_A and η_B such that η_A and $\eta_A \cup \eta_B$ are bases of A and $A \oplus B$, respectively.*

(b) *If $x_A \in [\eta_A]$ and $x_B \in [\eta_B]$, then (h_N denotes the p-height in N)*

$$h_N(x_A + x_B) = \min\{h_N(x_A), h_N(x_B)\}.$$

Then, if we write $\eta_B = \{y_i : i < k\}$, $k = |\eta_B|$, there is in $m(Q)$ a $k \times k$ diagonal invertible integral matrix $\delta = \{d_i : i < k\}$ such that the following conditions are satisfied.

(c) *$\tau = \Pi_B(\delta\eta_B)$ is an ordered basis of B. (Here, Π_B is the canonical projection of N onto B.)*

(d) *There is a height-preserving isomorphism ρ from $[\delta\eta_B]$ onto $[\tau]$ such that $\rho(\alpha\delta\eta_B) = \alpha\tau$ for all $\alpha \in R^k$.*

LEMMA 3. *Let k be the rank of M and M'. Let $\eta = \{y_i : i < k\}$ and $\eta' = \{y'_i : i < k\}$ be ordered bases of M and M', respectively, with η' summandwise (that is, each $y'_i \in M_j$ for a j). If J is a countable subset of I, then there is a set T such that the following conditions are satisfied.*

(a) *T is countable and $J \subseteq T \subseteq I$.*

(b) *Define* $\eta(T)=\{y_i\in\eta : y_i\in M(T)\}$. $\eta(T)$ *and* $\eta'(T)$ *are bases of* $M(T)$ *and* $M'(T)$, *respectively.*

(c) $y_i\in\eta(T)$ *if and only if* $y_i'\in\eta'(T)$.

LEMMA 4. *M and M' have the same basis type if and only if there is a partition of I into countable subsets $\{I_j : j<I\}$ such that $M(I_j)$ and $M'(I_j)$ have the same basis type for each index $j<I$.*

Using Lemma 2, 3 and a transfinite induction, we can prove Lemma 4. Theorem 2 is immediate from Lemma 4.

COROLLARY. *$M\simeq M'$ if and only if they have isomorphic torsion parts and contain basic free submodules F and F' of, respectively, with a height-preserving isomorphism from F onto F'.*

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