A CLASSIFICATION OF MODULES OVER COMPLETE DISCRETE VALUATION RINGS

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1. Introduction. The purpose of this paper is to announce the completion of a classification (up to isomorphism) of all modules which are direct sums of countably generated modules over complete discrete valuation rings. The detailed proofs will appear elsewhere. Throughout this paper, let $R$ denote a fixed but arbitrary complete discrete valuation ring and $p$ a fixed but arbitrary prime element of $R$. For the sake of convenience, a cardinal is viewed as the first ordinal having that cardinality. Let $(c, R, k)$ be the class of all countably generated reduced $R$-modules of (torsion-free) rank $\leq k$ and $D(c, R, k)$ that of all direct sums of members of $(c, R, k)$. Clearly

$$(c, R, 0) \subset (c, R, 1) \subset \cdots \subset (c, R, \omega)$$

$D(c, R, 0) \subset D(c, R, 1) \subset \cdots \subset D(c, R, \omega).$

Notice that a $p$-primary abelian group is a member of $(c, R, 0)$, particularly if $R$ is a ring of $p$-adic integers. A classification (of all members) of $(c, R, k)$ was done by Ulm (1933) when $k=0$ [8], by Kaplansky and Mackey (1951) when $k=1$ [4], by Rotman and Yen (1961) when $k<\omega$ [7], and that of $D(c, R, k)$ was done by Kolettis (1960) when $k=0$ [5]. First, we complete a classification of $(c, R, \omega)$ and then, utilizing this, we finish that of $D(c, R, \omega)$.

2. Invariants. We need two kinds of invariants, namely, the Ulm invariants and the basis types. Since the celebrated Ulm invariants are well known, a brief explanation of the basis types only is in order [2], [4], [7]. Let $R^k = \bigoplus \{ R : i < k \}$ for each $k$. Define $f(R)$ to be the class of all sordinal (ordinal or $\omega$) valued functions on $R^k$ for all cardinals $k$, and $m(Q)$ that of all square row-finite matrices over $Q$, the quotient field of $R$. Suppose that $f, g \in f(R)$. Define $f \sim g$ to mean

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both that Dom \( f = \text{Dom} \ g = R^k \) for some cardinal \( k \) and that there is a matrix \( \gamma \) and a diagonal matrix \( \delta \), both \( k \times k \) invertible integral (that is, all entries are elements of \( R \)) in \( m(Q) \), such that \( f(\alpha \gamma) = g(\alpha \delta) \) for all \( \alpha \in R^k \). It is routine to show that \( \sim \) is an equivalence relation on \( f(R) \).

Let \( M \) be an \( R \)-module of rank \( k \). Then, every basis \( \eta = \{ y_i : i < k \} \) defines a function \( g \) of \( f(R) \) by

\[
g(\alpha) = h_p(\alpha \eta) = h_p(\sum a_i y_i : i < k)\]

for all \( \alpha = \{ a_i : i < k \} \in R^k \). Notice that \( g = \infty \) if \( k = 0 \) since a sum without term is 0. It is routine to show that \( g \sim g' \) if \( g' \) is defined by another basis of \( M \). Thus, \( M \) determines uniquely a class of \( f(R)/\sim \), which we call the basis type of \( M \). It is easy to show the following lemma.

**Lemma 1.** Two reduced \( R \)-modules \( M \) and \( M' \) have the same basis type if and only if they contain basic free submodules \( F \) and \( F' \), respectively, with a height-preserving isomorphism from \( F \) onto \( F' \).

3. A classification of \((c, R, \omega)\).

**Theorem 1.** Let \( M \) and \( M' \) be countably generated reduced \( R \)-modules. Then, \( M \sim M' \) if and only if they have the same Ulm invariant and the same basis type.

Only the "if" part needs a proof. Let \( \alpha = \{ a_i : i < k \} \). Define \( \alpha(r) = \{ a_i : i < r \} \) for each number \( r \). Let \( k \) be the same rank of \( M \) and \( M' \). Then, by Lemma 1, there are ordered bases \( \eta = \{ y_i : i < k \} \) and \( \eta' = \{ y'_i : i < k \} \) of \( M \) and \( M' \), respectively, with a height-preserving isomorphism \( \rho \) such that \( \rho(\alpha \eta) = \alpha \eta' \) for all \( \alpha \in R^k \). We may assume that there are countable subsets \( \xi = \{ x_i : i < \omega \} \) and \( \xi' = \{ x'_i : i < \omega \} \) of \( M \) and \( M' \), respectively, such that

\[
M = [\xi \cup \eta] \quad \text{and} \quad M' = [\xi' \cup \eta'] \quad \text{with} \quad px_i \in [\xi(i) \cup \eta(i)] \quad \text{and} \quad px'_i \in [\xi'(i) \cup \eta'(i)] \quad \text{for each} \quad i < \omega.
\]

The main idea of the proof is to construct a sequence of height-preserving isomorphisms \( \{ \phi_i : i < \omega \} \) in such a way that the following conditions are satisfied.

(a) \( \phi_i : A_i \rightarrow A'_i \) where

\[
A_i = [\xi(i) \cup \eta(i) \cup \phi_i^{-1}(\xi'(i) \cup \eta'(i))], \quad A'_i = [\xi'(i) \cup \eta'(i) \cup \phi_i(\xi(i) \cup \eta(i))].
\]

(b) \( \phi_0 \leq \cdots \leq \phi_i \leq \phi_{i+1} \leq \cdots \)

(c) There exists a nonnegative integer \( n(i) \) such that \( p^n(i) A_i \)
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and \( p^{n(i)}A \) and \( \phi_i = \rho \) as height-preserving isomorphism from \( p^{n(i)}A \) onto \( p^{n(i)}A' \).

The supremum of \( \{ \phi_i : i < \omega \} \) gives the required isomorphism from \( M \) onto \( M' \). For more detailed proof, see [1] or [2].

4. A classification of \( D(c, R, \omega) \).

THEOREM 2. Let \( M \) and \( M' \) be direct sums of countably generated reduced \( R \)-modules. Then, \( M \cong M' \) if and only if they have the same Ulm invariant and the same basis type.

Again, only the “if” part needs a proof. We may write as

\[
M = \bigoplus \{ M_i : i \in I \} \quad \text{and} \quad M' = \bigoplus \{ M'_i : i \in I \}
\]

where all \( M_i, M'_i \in (c, R, \omega) \) and \( I \) is a cardinal. For notational convenience, define \( M(T) = \bigoplus \{ M_i : i \in T \}, T \subseteq I \). The main idea of the proof is to show that there is a partition of \( I \) into countable subsets \( \{ I_j : j < I \} \) such that, for each \( j < I \), \( M(I_j) \) and \( M'(I_j) \) have the same Ulm invariant and the same basis type. Then by Theorem 1, they are isomorphic and, consequently, \( M \cong M' \). In fact, by the Kolettis theorem [3], [5], [6], we may assume that \( M_i \) and \( M'_i \) have already the same Ulm invariant for each \( i \). The following lemmas indicate the route of the proof.

LEMMA 2. Let \( N = A \oplus B \oplus C \) be a reduced \( R \)-module such that the following conditions are satisfied.

(a) There are in \( N \) disjoint subsets \( \eta_A \) and \( \eta_B \) such that \( \eta_A \) and \( \eta_A \cup \eta_B \) are bases of \( A \) and \( A \oplus B \), respectively.

(b) If \( x_A \in [\eta_A] \) and \( x_B \in [\eta_B] \), then \( h_N(x_A + x_B) = \min \{ h_N(x_A), h_N(x_B) \} \).

Then, if we write \( \eta_B = \{ y_i : i < k \}, k = |\eta_B|, \) there is in \( m(Q) \) a \( k \times k \) diagonal invertible integral matrix \( \delta = \{ d_i : i < k \} \) such that the following conditions are satisfied.

(c) \( \tau = \Pi_B(\delta \eta_B) \) is an ordered basis of \( B \). (Here, \( \Pi_B \) is the canonical projection of \( N \) onto \( B \).)

(d) There is a height-preserving isomorphism \( \rho \) from \([\delta \eta_B]\) onto \([\tau]\) such that \( \rho(\alpha \delta \eta_B) = \alpha \tau \) for all \( \alpha \in R^k \).

LEMMA 3. Let \( k \) be the rank of \( M \) and \( M' \). Let \( \eta = \{ y_i : i < k \} \) and \( \eta' = \{ y'_i : i < k \} \) be ordered bases of \( M \) and \( M' \), respectively, with \( \eta' \) summandwise (that is, each \( y'_i \in M_i \) for \( a j \). If \( J \) is a countable subset of \( I \), then there is a set \( T \) such that the following conditions are satisfied.

(a) \( T \) is countable and \( J \subseteq T \subseteq I \).
(b) Define \( \eta(T) = \{ y_i \in \eta : y_i \in M(T) \} \). \( \eta(T) \) and \( \eta'(T) \) are bases of \( M(T) \) and \( M'(T) \), respectively.

(c) \( y_i \in \eta(T) \) if and only if \( y_i \in \eta'(T) \).

**Lemma 4.** \( M \) and \( M' \) have the same basis type if and only if there is a partition of \( I \) into countable subsets \( \{ I_j : j < I \} \) such that \( M(I_j) \) and \( M'(I_j) \) have the same basis type for each index \( j < I \).

Using Lemma 2, 3 and a transfinite induction, we can prove Lemma 4. Theorem 2 is immediate from Lemma 4.

**Corollary.** \( M \cong M' \) if and only if they have isomorphic torsion parts and contain basic free submodules \( F \) and \( F' \) of, respectively, with a height-preserving isomorphism from \( F \) onto \( F' \).

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**Bibliography**


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