

FOUR CLASSES OF SEPARABLE METRIC INFINITE-DIMENSIONAL MANIFOLDS¹

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Communicated by R. D. Anderson, October 6, 1969

1. Introduction. The purpose of this note is to announce some new embedding, homeomorphism, and characterization theorems regarding certain infinite-dimensional manifolds. We list these theorems below along with some of the principal known results in this area. It is expected that these new results will constitute a portion of the author's dissertation and their proofs will appear in a longer paper that is in preparation.

2. Definitions and notation. Each infinite-dimensional separable Fréchet space (and therefore each infinite-dimensional separable Banach space) is homeomorphic to s , the countable infinite product of open intervals $(-1, 1)$ (see [3]). A *Fréchet manifold* (or *F-manifold*) is a separable metric manifold modeled on s . A *Hilbert cube manifold* (or *Q-manifold*) is a separable metric manifold modeled on the Hilbert cube I^∞ , which we represent as the countable infinite product of closed intervals $[-1, 1]$.

Let σ be the set consisting of all points in s having at most finitely many nonzero coordinates and define a σ -*manifold* to be a separable metric manifold modeled on σ . Let Σ be the set consisting of all points in s having at most finitely many coordinates not in $[-\frac{1}{2}, \frac{1}{2}]$ and define a Σ -*manifold* to be a separable metric manifold modeled on Σ .

A subset K of a space X is a *Z-set in X* if K is closed and if for every nonnull homotopically trivial open set U in X , $U \setminus K$ is nonnull and homotopically trivial.

A subset M of a metric space X is said to have the (*finite-dimensional*) *compact absorption property* (or ($f-d$) cap) in X provided that

(1) $M = \bigcup_{n=1}^{\infty} M_n$, where M_n is a (finite-dimensional) compact Z-set in X , and

AMS Subject Classifications. Primary 5755; Secondary 5705, 5701.

Key Words and Phrases. Fréchet manifolds, Hilbert cube manifolds, Property Z, (finite-dimensional) compact absorption property, infinite deficiency.

¹ This article was presented to the Society January 24, 1970.

The research was conducted while the author held a NASA traineeship.

(2) for each (finite-dimensional) compact set K in X , each integer $m > 0$, and each $\epsilon > 0$, there is an integer $n > 0$ and an embedding $h: K \rightarrow M_n$ such that $h|_{K \cap M_m} = \text{id}$ and $d(h, \text{id}) < \epsilon$.

An $(f-d)$ cap-set for X is a set which has the $(f-d)$ cap in X . We adopt the convention that the symbol $(f-d)$ cap implies two alternative conditions, one for $f-d$ cap and the other for cap. In [2] it is shown that σ is an $f-d$ cap-set for s and Σ is a cap-set for s .

Let X be a space and let \mathcal{U} be an open cover of X . A function $f: X \rightarrow X$ is said to be *limited by* \mathcal{U} provided that for each $x \in X$, x and $f(x)$ are both contained in some member of \mathcal{U} . By $\text{St}^n(\mathcal{U})$ we will mean the n th star of the cover \mathcal{U} .

As in [6], a subset K of a space X is said to be *strongly negligible* provided that given any open cover \mathcal{U} of X , there is a homeomorphism of X onto $X \setminus K$ which is limited by \mathcal{U} .

A subset K of I^∞ is said to have infinite deficiency (or *infinite codimension*) if for each of infinitely many different coordinate directions, K projects onto a single interior point of $(-1, 1)$.

3. Principal results on F , Q , Σ , and σ -manifolds. The theorems are those of the author unless other authorship is denoted.

I. CHARACTERIZATION OF MANIFOLDS BY HOMOTOPY TYPE.

THEOREM 1 (HENDERSON [8]). *If X and Y are F -manifolds of the same homotopy type, then they are homeomorphic.*

THEOREM 2. *If X and Y are both σ -manifolds or both Σ -manifolds and are of the same homotopy type, then they are homeomorphic.*

II. OPEN EMBEDDING THEOREMS.

THEOREM 3 (HENDERSON [8]). *If X is an F -manifold, then X can be embedded as an open subset of s .*

THEOREM 4. *If X is a σ (or Σ)-manifold, then X can be embedded as an open subset of σ (or Σ).*

III. PRODUCT THEOREMS.

THEOREM 5 (WEST [9]). *If K is any countable locally-finite simplicial complex, then $|K| \times s$ is an F -manifold and $|K| \times I^\infty$ is a Q -manifold.*

THEOREM 6. *If K is any countable locally-finite simplicial complex, then $|K| \times \sigma$ is a σ -manifold and $|K| \times \Sigma$ is a Σ -manifold.*

THEOREM 7. *If X is a σ -manifold and Y is a Σ -manifold of the same homotopy type, then $X \times I^\infty$ and Y are homeomorphic.*

IV. FACTOR THEOREMS.

THEOREM 8 (HENDERSON [8]). *If X is an F -manifold, then there is a countable locally-finite simplicial complex K such that X and $|K| \times s$ are homeomorphic.*

THEOREM 9. *If X is a σ (or Σ)-manifold, then there is a countable locally-finite simplicial complex K such that X is homeomorphic to $|K| \times \sigma$ (or $|K| \times \Sigma$).*

THEOREM 10 (ANDERSON AND SCHORI [5]). *If X is an F -manifold, then X , $X \times s$, and $X \times I^\infty$ are all homeomorphic.*

THEOREM 11 (ANDERSON AND SCHORI [5]). *If X is a Q -manifold, then X and $X \times I^\infty$ are homeomorphic.*

THEOREM 12. *If X is a σ -manifold and I^n is any n -cell, then X , $X \times \sigma$, and $X \times I^n$ are all homeomorphic.*

THEOREM 13. *If X is any Σ -manifold, then X , $X \times \Sigma$, and $X \times I^\infty$ are all homeomorphic.*

V. RELATIONSHIPS BETWEEN F , Q , Σ , AND σ -MANIFOLDS.

THEOREM 14. *If X is a σ (or Σ -manifold), then X can be embedded as an $f-d$ cap (or cap)-set for an F -manifold and also for a Q -manifold.*

THEOREM 15 (ANDERSON [2]). *If X is I^∞ or s and M, N are $(f-d)$ cap-sets for X , then there is a homeomorphism of X onto itself taking M onto N .*

THEOREM 16. *If X is an F or Q -manifold and M is an $f-d$ cap (or cap)-set for X , then M is a σ (or Σ)-manifold.*

THEOREM 17. *If X is an F or Q -manifold, M and N are $(f-d)$ cap-sets for X , and \mathfrak{U} is an open cover of X , then there is a homeomorphism of X onto itself which takes M onto N and is limited by \mathfrak{U} .*

THEOREM 18 (ANDERSON [2]). *If M is an $(f-d)$ cap-set for I^∞ , then $I^\infty \setminus M$ is homeomorphic to s .*

THEOREM 19. *If X is any Q -manifold and M is an $(f-d)$ cap-set for X , then $X \setminus M$ is an F -manifold which is of the same homotopy type as X .*

VI. SUBSETS AND SUPERSETS OF F , Q , Σ , AND σ -MANIFOLDS.

THEOREM 20 (ANDERSON, HENDERSON AND WEST [6]). *A necessary and sufficient condition that a closed subset K of an F -manifold be a Z -set is that K be strongly negligible.*

THEOREM 21. *A necessary and sufficient condition that a closed subset K of a σ or Σ -manifold be a Z -set is that K be strongly negligible.*

THEOREM 22. *Let M be an $(f-d)$ cap-set for an F or Q -manifold X and let K be a Z -set in X . Then $M \setminus K$ is an $(f-d)$ cap-set for X .*

THEOREM 23. *Let M be an $(f-d)$ cap-set for an F or Q -manifold X and let K be a countable union of (finite-dimensional) compact Z -sets in X . Then $M \cup K$ is an $(f-d)$ cap-set for X .*

VII. HOMEOMORPHISM EXTENSION THEOREMS.

THEOREM 24 (ANDERSON AND McCHAREN [4]). *Let X be an F -manifold, let K_1, K_2 be Z -sets in X , let \mathfrak{U} be an open cover of X , and let h be a homeomorphism of K_1 onto K_2 such that there is a homotopy $H: K_1 \times I \rightarrow X$ for which $H_0 = \text{id}$, $H_1 = h$, and $H(\{x\} \times I)$ is contained in some member of \mathfrak{U} , for each $x \in K_1$. Then h can be extended to a manifold homeomorphism which is limited by $\text{St}^4(\mathfrak{U})$.*

THEOREM 25. *Let X be a σ or Σ -manifold, let K_1, K_2 be Z -sets in X , let \mathfrak{U} be an open cover of X , and let h be a homeomorphism of K_1 onto K_2 such that there is a homotopy $H: K_1 \times I \rightarrow X$ for which $H_0 = \text{id}$, $H_1 = h$, and $H(\{x\} \times I)$ is contained in some member of \mathfrak{U} , for each $x \in K_1$. Then h can be extended to a manifold homeomorphism which is limited by $\text{St}^{28}(\mathfrak{U})$.*

VIII. COMPLETE EXTENSIONS OF Σ AND σ -MANIFOLDS.

THEOREM 26. *Let X be a σ (or Σ)-manifold and let Y be a complete separable metric space containing X . Then there is an F -manifold Z such that $X \subset Z \subset Y$ and X is an $f-d$ cap (or cap)-set for Z .*

IX. INFINITE DEFICIENCY.

THEOREM 27 (ANDERSON [1]). *Let X be I^∞ or s and let K be a closed subset of X . A necessary and sufficient condition that K be a Z -set in X is that there exists a homeomorphism of X onto itself taking K onto a set having infinite deficiency.*

THEOREM 28 (CHAPMAN [7]). *Let X be an F -manifold and let K be a closed subset of X . A necessary and sufficient condition that K be a Z -set in X is that there exists a homeomorphism h of X onto $X \times s$ such that $\pi_s \circ h(K)$ has infinite deficiency.*

THEOREM 29. *Let X be a Q -manifold and let K be a closed subset of X . A necessary and sufficient condition that K be a Z -set in X is that there exists a homeomorphism h of X onto $X \times I^\infty$ such that $\pi_{I^\infty} \circ h(K)$ has infinite deficiency.*

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