

## A NONSOLVABLE GROUP OF EXPONENT 5

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**THEOREM 1.** *There exists a group  $\mathfrak{G}$  of exponent 5 which is locally nilpotent, but not nilpotent. In particular,  $\mathfrak{G}$  is not solvable.*

Thus there exist varieties which are nonsolvable, but locally finite and locally solvable.

To prove Theorem 1, we first show that a certain ring is not nilpotent. Let  $R$  be the free associative ring of characteristic 5 generated by noncommuting indeterminates  $x_1, x_2, \dots$ , and let  $L$  be the Lie ring in  $R$  generated by  $x_1, x_2, \dots$  where addition in  $L$  is the same as in  $R$  and Lie multiplication is commutation  $[x, y] = xy - yx$  in  $R$ . An element of  $L$  will be called a Lie element.

**THEOREM 2.** *If we impose on  $R$  the following identical relations for Lie elements  $x$  and  $y$ :*

$$(i) \quad x^3 = 0$$

and

$$(ii) \quad x^2y - 3xyx + 3yx^2 = 0$$

*then the resulting ring is not nilpotent.*

**REMARK.** Higgins in [3] showed that (i) and (ii) holds in the endomorphism ring of the additive group of a Lie ring satisfying the third Engel condition.

Also worth mentioning is the following result which is equivalent to Theorem 2 as shown in Walkup [8].

**THEOREM 3.** *There exists a Lie ring of characteristic 5 which satisfies the third Engel condition and which is not nilpotent.*

G. Higman [4] and A. I. Kostrikin [5] showed that a Lie ring of characteristic 5 satisfying the fourth Engel condition is locally nilpotent, and in view of Theorem 3, this is the best one can say.

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Kostrikin [6], in fact, was able to prove the very general theorem that a Lie ring satisfying the  $n$ th Engel condition and having prime characteristic  $p > n$  is locally nilpotent.

In getting away from the finite generation condition, P. Higgins [3] and Heineken [2], showed that an associative ring with characteristic prime to 2, 3, 5 and 7 in which the cube of every Lie element is zero must be nilpotent of index at most  $3^9$ . D. Walkup [8] in his thesis improved this result in two ways. First he showed that no restriction on the prime 7 is necessary and secondly that the nilpotency index can be greatly reduced. Specifically he showed

**THEOREM 4.** *Let  $R'$  be the free associative ring generated by noncommuting indeterminates  $x_1, x_2, \dots$ , with coefficients in a ring in which division by 2, 3, and 5 are possible and the cubes of all Lie elements are zero. Then  $R'$  is nilpotent of index at most 9.*

We sketch the proof of Theorem 2 and the deduction of Theorem 1 from it.

Relations (i) and (ii) together are equivalent to the "Higgins relations",

$$(iii) \quad xyz + xzy = yzx + zyx = 2(yxz + zxy)$$

for all (homogeneous) Lie elements  $x, y, z$ .

Let  $H$  be the ideal of  $R$  generated by (iii). Denote by  $R_n$  the vector subspace of  $R$  with basis consisting of all monomials of total degree  $2n$  and degree 2 in each indeterminate  $x_i, i=1, 2, \dots, n$ . Making use of the relations (iii), we are able to establish inductively for each  $n \geq 2$  the existence of a linear transformation  $\alpha$  of  $R_n$  onto  $Z_5$ , the integers modulo five, which satisfies:

(a)  $\alpha(x_1^2 x_2^2 \cdots x_n^2) = 1$ .

(b)  $\alpha(MN) = \alpha(NM)$ , where  $M$  and  $N$  are any monomials such that  $MN$  is in  $R_n$ .

(c)  $\alpha(M) = \alpha(M')$ , where  $M$  is any monomial in  $R_n$  and  $M'$  is the monomial obtained from it by permuting (the names of) the indeterminates.

(d)  $\alpha(M) = \alpha(M^T)$ , where  $M$  is any monomial in  $R_n$  and  $M^T$  is the monomial obtained from it by reversing the order of the factors (of degree 1).

(e)  $\alpha[M(xyz + xzy)] = \alpha[M(yzx + zyx)] = 2\alpha[M(yxz + zxy)]$ , where  $x, y$ , and  $z$  are chosen from among the generators  $x_i$  and  $M$  is any monomial such that the indicated products are in  $R_n$ .

We then show that the kernel of  $\alpha$ , say  $S_n$ , contains  $H \cap R_n$ , i.e., (e) holds for all Lie elements  $x, y$  and  $z$  such that  $Mxyz$  is in  $R_n$ .

Since  $x_1^2 x_2^2 \cdots x_n^2$  is not in  $S_n$  and hence not in  $H$ ,  $R$  is *not* nilpotent modulo  $H$ , establishing Theorem 2.

Using Bruck's notation in §3 of [1], we can show that Theorem 2 implies that  $R$  is not nilpotent modulo the permutation ideal of  $T_k$ . Then, by Theorem 4.3 of [1] this last fact implies the negation of the statement  $R(3, \pi)$  in Bruck's notes [1], i.e.,

**THEOREM 5.** *There exists a group ring  $Z_5G$  over the field  $Z_5$  of integers modulo 5 such that the augmentation ideal of  $Z_5G$  is not nilpotent modulo the ideal  $I$  generated by all elements  $(g-1)^3$  with  $g$  in  $G$ .*

To complete the proof of Theorem 1, we use a standard construction. Let  $Z_5G$  and  $I$  be as in Theorem 5. Define group  $\mathfrak{G}$  to be the set of all ordered pairs  $\{g, r\}$ ,  $g \in G$ ,  $r \in Z_5G/I$  with the multiplication

$$\{g, r\} \{h, s\} = \{gh, rh + s\}.$$

An easy check shows that  $\mathfrak{G}$  has exponent 5. If  $a = \{1, 1\}$  and  $b_i = \{g_i, 0\}$ , then the commutator

$$(a, b_1, b_2, \dots, b_n) = \{1, (g_1 - 1)(g_2 - 1) \cdots (g_n - 1)\}.$$

Since the augmentation ideal of  $Z_5G$  is not nilpotent modulo  $I$ ,  $\mathfrak{G}$  is not nilpotent, and hence by a theorem of Tobin [7],  $\mathfrak{G}$  is not solvable. Thus,  $G$  is also nonsolvable.

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