constructed via something like a projective limit from a totally ordered compact set X and for each $x \in X$ a cylindrical semigroup S_x and maps $\mathfrak{M}_{xy}: S_y \to S_x$ for $x \leqq y$ which satisfies various properties. Then they show compact S is irreducible iff S/H is an I-semigroup and S is irreducible iff S is a hormos with trivial group of units and some additional conditions on $\mathfrak{M}_{xy}$'s. This clears up the structure of irreducible semigroups, a result due to the authors, and appearing in print for the first time here. The authors sum up all these results of Chapter B and call it the "Second fundamental theorem of compact semigroups."

The next chapter, Chapter C, applies the previous results to some rather special cases. The final chapter (before three appendices) contains 42 pages of examples with pictures, many due to Hunter and Anderson.

All in all Hofmann and Mostert have given us an important and successful book.

We have come to the end.

You lovers of semigroups can run out and purchase Clifford and Preston's book or Hofmann and Mostert's book. For the rest of you I leave you with the advice of Rousseau's deformed (one nippled) mistress, "Zanetto, lascia le donne, et studia la mathematica."

JOHN RHODES

Statistical Mechanics by David Ruelle, Benjamin, New York, 1969. $11+219$ pp. \$7.95 paper, \$17.00 cloth.

Statistical Mechanics used to be exclusively a branch of physics. In the last fifteen years it has become a recognizable part of mathematics. Ruelle's book is one of the first to bring out clearly the intrinsic mathematical content of the subject.

Most textbooks on statistical mechanics written by and for physicists tend to be carbon copies of each other. They traditionally begin with an introductory chapter pretending to lay the foundations of the subject, but usually concluding that the outstanding problem in this line is the ergodic conjecture, which is barely discussed and promptly discarded as in the domain of mathematicians. Such a book then proceeds to deal with systems of noninteracting particles, or occasionally with crude approximations for interacting systems, whose justification is again based more on common—and tacit—consent than on even the rudiments of an acceptable proof. While it is true that considerable progress has been made on the ergodic conjecture (which is not touched upon at all in the present book), notably by Sinai, it is grotesquely incorrect to go on believing, in this day and age, that there are no other important mathematical problems in the subject.

Mathematical statistical mechanics is not—or at least, not yet—a conventional branch of probability or combinatorial theory. Most problems in the subject are indeed probabilistic and combinatorial, but no clues to their solution are to be found in the conventional mathematical literature. Most of what has been accomplished in the past fifteen years has been done by inventing new techniques and points of view suitable for handling systems with an infinite number of degrees of freedom. The one mathematics book that has proved really useful in this enterprise is Hardy, Littlewood and Pólya's book, "Inequalities." Most conventional probability and combinatorial theory deals with one-dimensional systems—that is to say, with systems that in some vaguely defined sense can be ordered. Unfortunately, the problems of statistical mechanics are intrinsically multi-dimensional.

The subfield of statistical mechanics that has seen the most progress is equilibrium statistical mechanics, and Ruelle's book is almost exclusively devoted to it. As in most of mathematical physics, one must eschew the temptation of overproving the obvious. From the point of view of conventional physics wisdom a great many of Ruelle's theorems would fall in that category but the viewpoint of physicists is beginning to change; mathematicians, however, will have no difficulty appreciating the beauty and depth of what is given in these pages. In any case, modern statistical mechanics is really quite young and, as is the case with every field, must have a beginning. The book should therefore be judged according to how well it records what has been achieved to date and this it does admirably.

Ruelle begins with a treatment of the thermodynamic limit for infinite systems. He chooses, perhaps somewhat artificially, to further subdivide the subject into a treatment of lattice systems (Chapter 2), and one of continuous systems (Chapter 3). An introductory first chapter attempts to set the stage and give some preliminary definitions. It is so compact, however, that the beginner will probably find it inadequate, and the expert will consider it little more than a glossary. In Chapters 2 and 3 the reader quickly settles down to the meat and potatoes of the subject. Gibbs and others have told us that if one accepts the ergodic conjecture, then one may completely describe the equilibrium properties of a large system in terms of a phase space average called the partition function Z. What one wants to prove is that this function, which cannot be evaluated exactly for nontrivial systems, has those properties which would lead in the limit to physically correct consequences for macroscopic systems. In other words, would the phase space average, if correctly carried out, be in accord with the science of thermodynamics? The limit in question is the following: One allows the number of particles N to go to infinity and the volume V of the container to go to infinity in such a way that the ratio of the two, called the density, remains fixed. Apart from mathematically irrelevant constants, the logarithm of the partition function is called the free energy, and the question boils down to (a) deciding whether the free energy divided by the volume approaches a limiting function of the density and (b) whether this limiting function is convex as a function of the density. Some physicists regard this question as academic, because if the answer were to be negative, then matter as we know it could not exist. Nevertheless, the mathematics of this question is far from trivial. A system of particles (such as a galaxy) interacting through purely attractive gravitational forces cannot have a "nice" thermodynamic limit, because such a system would collapse. Contrariwise, a system of particles interacting only through purely repulsive electrical forces cannot have a "nice" limiting thermodynamics, because such a system would explode. There are even more startling examples. On page 36 Ruelle displays a system with a harmless-looking interaction potential of finite range (compact support) which he shows, despite wishful thinking to the contrary, does not have an acceptable thermodynamic limit.

Up to the time the book was written a method had evolved for proving the existence of the thermodynamic limit for systems with interaction potentials that vanish sufficiently rapidly at infinity and that are sufficiently repulsive at short distances. The proof involves an ingenious argument based on subdividing the volume. It now seems simple, but it took ten years to develop. In retrospect, one may say that the pieces of the proof were known but never put together. It is a bit like finding the bones of a dinosaur in the desert. Each piece is simple but they go together in only one way. In order to find that way it was necessary to develop the appropriate perspective, and such perspective may well prove useful in other problems of statistical mechanics.

Nevertheless, the problem remained open of proving the existence of the limit for systems without short range forces, notably with long-range Coulomb forces as exist among the real particles of nature. As mentioned above, such a system would explode were it not for the fact that one always has equal amounts of positive and negative charges so that in some sense a cancellation occurs and the net interaction behaves as though it were short range. Since the days of Debye and Hückel, it was well understood in detail how this screening, as it is called, actually takes place but, as we know from other fields of physics and mathematics, it is one thing to assume that the system is well behaved and calculate the details of its behavior on that basis, and quite another thing to prove that the system is indeed well mannered and that there are no corners of phase space containing the seeds of instability. The proof for Coulomb systems was finally achieved just before the book went to press and a note to that effect may be found on page 64.

With the bulk properties in a satisfactory state the next question that arises is the existence of correlation functions. If we define $\rho(x_1, x_2)$ to be the probability density that two particles are to be found at positions x_1 and x_2, does this function have a limit as N and V go to infinity? In almost all approximate calculations that are made in the physics literature, one assumes that such a limiting function exists and goes on to calculate its properties. But its existence is far from established. Chapter 4, dealing with low density expansions and correlation functions, describes what little is known about this question. Firstly, one can show that if the density is low enough, then a power series expansion for the free energy per unit volume converges. The existence of such expansions was known from approximately the beginning of the Second World War, but the proof of their convergence is only about ten years old. The individual terms in such an expansion are given by complicated integrals, and these may be represented conveniently by graphs. The study of such graphs is an active field at present and has led to beautiful and important theorems. Unfortunately, none of this material is discussed in Ruelle's book, though it would be a delight to mathematicians with a combinatorial bent. Perhaps this is because the author feels that further progress will be achieved by going beyond such classical graph-theoretic methods.

Secondly, Chapter 4 shows how to prove the existence of the limiting correlation functions by means of power series expansions. The proof involves the techniques of Banach algebras, and the author admirably succeeds in displaying the power of functional analysis. Here is a brilliant counterexample to the oft-repeated dictum that modern mathematics is irrelevant to mathematical physics. Nevertheless, the existence of correlation functions generally is still left as an open question. The region of greatest physical interest is where power series expansions definitely do not converge, and here is a renewed challenge to modern mathematics to show its power again.

Chapter 5 is devoted to the problem of phase transitions. At this point all parties will agree that the book touches upon the core problem of statistical mechanics. The chapter only scratches the subject, but a proper discussion of what we know at present about phase transitions—even what we know about them rigorously—would require a book in itself. This is because a good deal of our knowledge stems from exact solutions, that is, from an evaluation in closed form of the partition function of certain model problems. Some of these problems are quite close to problems in combinatorial theory, such as evaluating the number of three-colorings of an infinite checkerboard, but to discuss these solutions would require a considerable combinatorial detour. What Ruelle does do instead is to bring under one roof several general theorems about model systems that do have phase transitions, even though we cannot always evaluate the partition functions of those models exactly. For the ferromagnetic Ising model, the beautiful circle theorem of Lee and Yang and the important inequalities of Griffiths, Sherman and Kelly are elegantly presented. Very recently, however, Ginibre has found a two-line proof of Griffiths' inequalities, thereby rendering this subsection obsolete. There is also a discussion of the Mermin and Wagner theorem, very important because it shows that phase transitions cannot occur in certain systems possessing a great deal of symmetry. The theorem is presented in bare mathematical form and little discussion is given of the underlying physical idea which motivated it. As in the case of the existence of the thermodynamic limit, the underlying physical ideas were known some time prior to the actual proof. Presenting the proof without the physical background makes the consequences of the theorem appear like black magic.

The question of phase transitions may be stated thus: The limiting free energy as a function of the density and other parameters such as the temperature is in general analytic. But for most systems of interest there are points or regions where the function is not analytic. The problem is to identify the varieties of singularities that can occur at these transition points. It is by no means obvious when we look at the partition function of a system of relatively weakly interacting particles, such as water molecules, that an infinitesimal change in density may bring about a major change in the derivative of the free energy. Physically, the change manifests itself as the condensation of steam to water or water to ice or ice I to ice II. We do know that the kinds of singularities that can occur run all the way from discontinuities in the first derivative to essential singularities where all derivatives are bounded and continuous on either side of the critical point, but where a power series expansion about the critical point diverges. Up to now, progress in this field has been made by heavy applications of classical analysis, and it remains to be seen whether modern mathematical ideas have any relevance.

The last two chapters of the book are devoted to a modern development, namely, the identification of the states of a physical system by positive linear functionals on a suitable B^*-algebra. The translational, rotational and other invariances of the system manifest themselves as groups of automorphisms of the B^*-algebra. The great hope behind such an approach is to be able to avoid the problem of starting with a finite number of particles and tediously passing to the limit of an infinite system. The B^*-algebra approach to statistical mechanics does not at present meet with favor in all quarters, and it has a diffuse reputation of being arcane. It is another attempt to apply functional analysis to classical problems in physics. As Ruelle himself says, "the results are largely due to physicists with a background of axiomatic, relativistic, quantum field theory. This imparts a somewhat special flavor to the subject." To be candid, we must admit that so far no major breakthroughs in statistical mechanics have been achieved by such methods nor, on the other hand, has a dead end been reached. The next decade will perhaps decide the issue.

As I said in the beginning, I consider this work a milestone in statistical mechanics and in mathematics. It is also a work of considerable scholarship. The author has taken pains to be right and thorough in his bibliographical references. The book is not easy to swim through, however. Mathematicians will feel comfortable with the terse style, but they will probably be at a loss to understand the physical background and motivation of the material. Physicists, on the other hand, even specialists, may find the style terse almost to the point of opaqueness. The situation might have been alleviated to a considerable extent by providing a glossary of symbols and notation, but this has not been done. The hard work, both on the part of the author and on the part of the series reader, will be well repaid by the opening up of new and fruitful vistas in mathematical inquiry.

ELLIOTT LIEB

Markov Processes with Stationary Transition Probabilities, by Kai Lai Chung, Springer-Verlag, Berlin, Heidelberg, New York, x+301 pp. $14.00

Markov processes, whose definition goes back to Markov (1907)