A SELF-UNIVERSAL CRUMPLED CUBE
WHICH IS NOT UNIVERSAL

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C. E. Burgess and J. W. Cannon [2, §10] have asked whether each self-universal crumpled cube is universal. In this note we give a negative answer to their question by showing that the familiar solid Alexander horned sphere $K$ is not universal. Casler has shown that $K$ is self-universal [3].

A crumpled cube $C$ is a space homeomorphic to the union of a 2-sphere $S$ topologically embedded in the 3-sphere $S^3$ and one of its complementary domains. The boundary of $C$, denoted $\text{Bd } C$, is the image of $S$ under the homeomorphism. A sewing $h$ of two crumpled cubes $C$ and $C^*$ is a homeomorphism of $\text{Bd } C$ to $\text{Bd } C^*$. The space $C \cup h C^*$ given by a sewing $h$ is the identification space obtained from the (disjoint) union of $C$ and $C^*$ by identifying each point $p$ in $\text{Bd } C$ with $h(p)$ in $\text{Bd } C^*$.

A crumpled cube $C$ is universal if, for each crumpled cube $C^*$ and each sewing $h$ of $C$ and $C^*$, the space $C \cup h C^*$ is topologically equivalent to $S^3$. Similarly, a crumpled cube $C$ is self-universal if $C \cup f C = S^3$ for each sewing $f$ of $C$ to itself.

1. A bad sewing. In order to define the desired sewing of the solid Alexander horned sphere $K$ to another crumpled cube $K^*$, we describe an upper semicontinuous decomposition of $S^3$ into points and almost tame arcs.

Let $H_1$ and $H_2$ denote the upper and lower half spaces of $E^3$, and $P$ the $xy$-plane. Let $A_0$ denote a solid double torus embedded in $E^3$ as shown in Figure 1 such that $A_0$ intersects $P$ in two disks $D_1$ and $D_2$. Letting $T_1$ and $T_2$ denote solid double tori embedded in $A_0$ as shown in Figure 1, we define $A_1$ as $T_1 \cup T_2$. Assuming sets $A_0, A_1, \ldots, A_n$ have been defined, let $A_n$ be the union of $2^n$ solid double tori contained in $A_{n-1}$ such that each double torus $T$ of $A_{n-1}$ contains exactly two components of $A_n$, which are embedded in $T$ just as $T_1$ and $T_2$ are embedded in $A_0$.

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Let $G'$ denote the upper semicontinuous decomposition of $E^3$ whose nondegenerate elements are the components of $\bigcap_{j=1}^{n} A_j$. By requiring that the components of $A_n$ become skinny as $n$ gets large, we force the nondegenerate elements of $G'$ to be arcs which are locally tame except at their lower end points.

With the addition of an ideal point $\infty$, $G'$ extends to a decomposition $G$ of $S^3$.

**Theorem 1.** The decomposition space $S^3/G$ is not homeomorphic to $S^3$.

The proof of Theorem 1 is discussed in the next section.

**Theorem 2.** Let $K$ denote the solid Alexander horned sphere. There exists a sewing $h$ of $K$ to a crumpled cube $K^*$ such that $K \cup_h K^*$ is not homeomorphic to $S^3$.

**Proof.** Let $\pi$ denote the natural projection of $S^3$ to $S^3/G$, and let $H_i^* = H_i \cup \{ \infty \}$ ($i = 1, 2$). Note that $\pi(H_i^*)$ is topologically equivalent
to $K$, and $\pi(H^*_2)$ is a crumpled cube $K^*$. The required sewing $h$ is the one induced by $\pi$ such that $K \cup hK^*$ and $S^3/G$ are homeomorphic.

REMARK. The procedure for defining $K^*$ is suggested by Stallings' crumpled cube [4].

2. Slicing homeomorphisms. Let $k$ be a nonnegative integer. A homeomorphism $h$ of $Bd A_0 \cup D_1 \cup D_2 \cup D_3$ into $A_0$ such that $h|Bd A_0 = \text{identity}$ is said to be slicing at stage $k$ if, for each solid double torus $T$ of $A_k$, each component of $T \cap h(D_i)$ ($i = 1$, $2$, $3$) is a disk embedded in $T$ just like a component of $T \cap P$.

A homeomorphism $h$ slicing at stage $k$ is said to satisfy Property $P_k$ if for some double torus $T$ of $A_k$ there exist components $X_1$, $X_2$, and $X_3$ of $T \cap h(\cup D_i)$ such that

\begin{enumerate}
  \item[(a)] $X_1 \cup X_2 \subset h(D_1 \cup D_2)$,
  \item[(b)] $X_3 \cap h(D_1 \cup D_2) = \emptyset$,
  \item[(c)] $X_3$ separates $X_1$ from $X_2$ in $T$.
\end{enumerate}

Theorem 1 is an immediate consequence of [1, Theorem 2] and the following lemmas.

**LEMMA 1.** If $h$ is a homeomorphism slicing at stages $k$ and $k + 1$ and satisfying Property $P_k$, then $h$ satisfies Property $P_{k+1}$.

**LEMMA 2.** If $h$ is a homeomorphism slicing at stage $k + 1$, then there exists a homeomorphism $h^*$ slicing at stages $k$ and $k + 1$ such that for each component $T$ of $A_{k+1}$, $T \cap h(D_i) = \emptyset$ implies $T \cap h^*(D_i) = \emptyset$ ($i = 1$, $2$, $3$).

**LEMMA 3.** Every homeomorphism slicing at stage $k$ satisfies Property $P_k$.

**LEMMA 4.** If there exists a nonnegative integer $k$ and a homeomorphism $g$ of $Bd A_0 \cup D_1 \cup D_2 \cup D_3$ into $A_0$ such that $g|Bd A_0 = \text{identity}$ and each component $T$ of $A_k$ intersects at most one of the disks $g(D_i)$, then there exists a homeomorphism $h$ slicing at stage $k$ that fails to satisfy Property $P_k$.

**References**


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