DIMENSION AND MULTIPlicity
FOR GRADED ALGEBRAS

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We want to reconsider a problem that goes back to Hilbert \[3\]. Let \( R = \sum R^p \) be a commutative algebra which is graded by the nonnegative integers and finitely generated over \( R^0 = F \), which for simplicity is a field. Let \( M = \sum M^p \) be a finitely generated graded \( R \)-module, with \( p \) again restricted to the nonnegative integers. Each component \( M^p \) is a finite-dimensional vector space over \( F \). If \( R \) is generated over \( F \) by elements homogeneous of degree one then Hilbert proved that there is a polynomial

\[
H_M(p) = e(M)\frac{p^{n-1}}{(n - 1)} + \cdots
\]

such that \( H_M(p) = \dim M^p \) for \( p \) large. With the understanding that the zero polynomial is of degree \(-1\), we may call \( n \) the dimension of \( M \). The coefficient \( e(M) \) is a nonnegative integer, the multiplicity of \( M \).

Unfortunately, if \( R \) is not generated by elements of degree one, it is not usually true that \( \dim M^p \) is eventually given by a polynomial in \( p \). (For example, let \( M = R = F[x] \) where \( x \) is an indeterminant of degree two.) The more general case, where the generators of \( R \) are of degree greater than one, arises naturally. We need a substitute for the Hilbert polynomial and it turns out that the Poincaré series

\[
P(M) = \sum (\dim M^p)t^p
\]

of the module is a good substitute. In the classical situation the relation between \( H_M \) and \( P(M) \) is such that \( H_M \) is of degree at most \( n - 1 \) if and only if \((1 - t)^nP(M)\) is a polynomial in \( t \). Moreover, if \( H_M \) is of degree exactly \( n - 1 \) then \( e(M) \) is the value of \((1 - t)^nP(M)\) for \( t = 1 \). We intend to show how these facts generalize. The details of the proofs will be given elsewhere.

In \[4\] Serre gave a homological treatment of dimension and multiplicity for local rings. Following Serre, we wish to define the multi-

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plicity of a graded module $M$ as an Euler characteristic of the complex
\[ \text{Tor}^R(F, M) = \sum \text{Tor}_i^R(F, M). \]

Let $C(R)$ be the category of all finitely generated graded modules over $R$, and all homomorphisms which are homogeneous of degree zero. Each $\text{Tor}_i^R(F, M)$ is a finite-dimensional graded vector space, a module of the category $C(F)$. As Fraser [2] has observed, it is natural to consider the Grothendieck groups $K(R)$ and $K(F)$ of the two categories, and attempt to define a multiplicity homomorphism $\chi_R: K(R) \to K(F)$. We set
\[ \chi_R(M) = \sum (-1)^i [\text{Tor}_i^R(F, M)] \]
where $[\text{Tor}^R(F, M)]$ is the image in $K(F)$. This makes sense if $\text{Tor}^R(F, M)$ is a finite complex. Surprisingly, the formula makes sense in the “completion” of $K(F)$ whether or not $\text{Tor}^R(F, M)$ is finite. Since a graded vector space $V$ is determined by the dimensions of its components, associating to $V$ its Poincaré polynomial $P(V)$ identifies $K(F)$ with the polynomial ring $\mathbb{Z}[t]$ over the integers. Using Eilenberg’s technique [1] of minimal resolutions it is easy to prove a lemma which insures that the above alternating sum is a well-defined formal power series in $t$.

**Lemma.** The $p$th component of $\text{Tor}_i^R(F, M)$ is zero if $p < i$.

From the long exact sequence for Tor we have a homomorphism $\chi_R: K(R) \to \mathbb{Z}[t]$ into the formal power series ring.

If every module in $C(R)$ has a finite resolution by free modules in $C(R)$, i.e., if $C(R)$ is of finite global dimension, then $\chi_R$ has values in the polynomial ring $\mathbb{Z}[t]$. In this case it is also true that $K(R)$ is a ring, with the product of two of the generators given by
\[ [M][N] = \sum (-1)^i [\text{Tor}_i^R(M, N)]. \]
This formula always makes sense in case one of the modules is free. The free modules of $C(R)$ are all of the form $R \otimes_F V$ for $V$ in $C(F)$. Thus in general $K(R)$ is a module over $K(F) = \mathbb{Z}[t]$.

**Theorem 1.** For any $R$, $\chi_R: K(R) \to \mathbb{Z}[t]$ is a homomorphism of $\mathbb{Z}[t]$-modules. If $C(R)$ has finite global dimension then $\chi_R: K(R) \to \mathbb{Z}[t]$ is a ring isomorphism.

Associate to a graded finite-dimensional vector space its total dimension. This yields a ring homomorphism $\text{dim}: \mathbb{Z}[t] \to \mathbb{Z}$ which is the natural augmentation, the function which assigns to a polynomial
its value for $t = 1$. If $C(R)$ is of finite global dimension then composing with $\chi_R$ gives a ring homomorphism $e_R: K(R) \to \mathbb{Z}$ and we have Serre's definition of the multiplicity in our situation:

$$e_R(M) = \sum (-1)^i \dim \text{Tor}^R_i(F, M).$$

The category $C(R)$ is of finite global dimension if (and probably only if) $R$ is a polynomial algebra $F[x_1, \ldots, x_n]$ generated by indeterminants which are homogeneous of positive degrees. In this case the Koszul complex can be used to compute multiplicities. Let $H_i(x, M)$ be the $i$th homology module of the Koszul complex of $x = (x_1, \ldots, x_n)$ and $M$.

**Theorem 2.** Let $R = F[x_1, \ldots, x_n]$ be a polynomial algebra generated by indeterminants of positive degrees $d_1, \ldots, d_n$. Then

$$\chi_R(M) = \sum (-1)^i [H_i(x, M)].$$

In particular, $\chi_R(F) = \prod (1 - t^{d_i})$.

In the classical situation the indeterminants are all of degree one, so $\chi_R(F) = (1 - t)^n$. This suggests the following theorem, which relates the multiplicity of a module to its Poincaré series.

**Theorem 3.** For any $R$ and any $M$ in $C(R)$, $\chi_R(M) = \chi_R(F)P(M)$.

**Corollary 1.** If $C(R)$ is of finite global dimension then $\chi_R(F)P(M)$ is a polynomial in $t$ and $e_R(M)$ is the value of this polynomial for $t = 1$.

We can always reduce to the case of finite global dimension by regarding $R$ as a quotient of a polynomial algebra $S$. An $R$-module $M$ becomes an $S$-module. The Poincaré series is unaffected, and $\chi_S(M)$ and $\chi_S(F)$ are polynomials.

**Corollary 2.** $\chi_R(M) = P(M)/P(R)$, and $P(M)$ and $\chi_R(M)$ are rational functions.

The relation $\chi_R(M) = P(M)/P(R)$ follows from the fact that $\chi_R(F)P(R) = \chi_R(R) = 1$.

**Corollary 3.** $\chi_R(M) = 0$ if and only if $M = (0)$.

Call $M$ of dimension at most $n$ if there are positive integers $d_1, \ldots, d_n$ such that $P(M)\prod (1 - t^{d_i})$ is a polynomial in $t$.

**Theorem 4.** The $R$-module $M$ is of dimension at most $n$ if and only if there exist homogeneous elements $y_1, \ldots, y_n$ in $R$ such that $M$ is finitely generated over the subalgebra $F[y_1, \ldots, y_n]$. 

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If \( P(M) \prod (1 - t^{d_i}) \) is a polynomial it is not true that we can always choose \( y_1, \ldots, y_n \) of degrees \( d_1, \ldots, d_n \). For example, let \( M = R = F[x, y] \) where \( x \) is an indeterminant of degree two and \( y \) is a non-zero element of degree one with \( y^2 = 0 \). The Poincaré series is

\[
P(M) = \frac{1 + t}{1 - t^2} = \frac{1}{1 - t}
\]

but \( R \) contains no element \( y_1 \) of degree one with \( M \) finitely generated over \( F[y_1] \).

**Corollary.** If \( R = F[y_1, \ldots, y_n] \) then every \( M \) in \( C(R) \) is of dimension at most \( n \).

**References**


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