ON PARALLELISM IN RIEMANNIAN MANIFOLDS

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The definition of parallelism along a curve in a Riemannian manifold extends to higher dimensional submanifolds. This note is to announce a local existence and uniqueness theorem, Theorem B(p), for the extended definition. A proof of the theorem in the $C^\infty$ category will appear in [2]. A proof, in the $C^0$ category, under somewhat weaker conditions, will appear in [1]. A global $C^\infty$ version under stronger assumptions appears in [3]. This note ends with a sketch of a new proof of Theorem B(p).

Let $g: N^p \to M^m$ be a (not necessarily isometric) smooth (that is, $C^\infty$ or $C^0$) immersion of Riemannian manifolds. Let $E$ be a euclidean vector bundle over $N$ and $F$ a euclidean vector bundle over $M$. A vector bundle map $G: E \to F$ is a vector bundle isometry along $g$ provided that $G$ sends the fibers $E(n)$ isometrically into the fibers $F(g(n))$. When $E$ and $F$ are the tangent bundles ($T(N^p)$ and $T(M^m)$), $G$ is called a tangent bundle isometry (T.B.I.) along $g$. The normal bundle to a T.B.I. $G$ is the $m-p$ dimensional vector bundle $G \perp$ over $N$ whose fiber over $n \in N$ is the orthogonal complement $\perp G(N_n)$ to $G(N_n)$ in $M_{g(n)}$. The second fundamental form of $G$, $\Pi_g: G \perp \to \text{Hom}(T(N), T(N))$ is a vector bundle map defined as follows. Given $v \in \perp G(N_n)$ and $x, y \in N_n$ extend $y$ to a vector field $Y$ on $N$ in some neighborhood of $n$, let $\nabla$ be the covariant derivation on $M$ and put

$$\langle \Pi_g(v)x, y \rangle_n = -\langle \nabla_{Tg(x)}G(Y), v \rangle_{g(n)}.$$  

The definition is independent of the choice of $Y$.

$G$ is parallel along $g$ if (trace)$ \cdot \Pi_g: G \perp \to \mathbb{R}$ vanishes identically. It was shown in [1] that this definition is a generalization to higher dimensional immersed submanifolds, of the classical notion of parallelism along a curve. The significant facts are the following.

Every unit vector field along a curve $g: N^1 = (a, b) \to M$ corresponds in a natural way to a T.B.I. along $g$. Under this correspondence, parallel vector fields are paired with parallel T.B.I.'s.

An immersion $g: N^p \to M^m$ is isometric if and only if its tangent map

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$Tg:TN\rightarrow TM$ is a T.B.I. In such a situation, $g$ is a minimal immersion if and only if $Tg$ is parallel along $g$. Thus for every $p, 1 \leq p < m$, the critical manifolds of the calculus of variations problem for minimal $p$ dimensional "area" are exactly the $p$ dimensional autoparallels (i.e. the isometric immersions whose tangent maps are parallel).

Below, the same letter is used to designate a distribution on a manifold and the subbundle of the tangent bundle that it determines. If $E$ is a vector bundle over $Y$ and $i:X\rightarrow Y$ is a smooth map then $i_*:i^*E\rightarrow E$ is the induced map of the induced bundle.

**Theorem B(p).** Let $g:N^p\rightarrow M^m$ be an (not necessarily isometric) immersion of Riemannian manifolds. Let $H$ be a $(p-1)$ dimensional distribution on $N^p$ and $(N^p, i)$ a homeomorphically embedded integral manifold of $H$. Suppose there is given as initial data:

1. $G^{p-1}:H\rightarrow T(M)$, a vector bundle isometry along $g$, and
2. $G^p:T(N^p)\rightarrow T(M)$, a vector bundle isometry along $g\cdot i$.

It is assumed that $G^{p-1}$ and $G^p$ are compatible:

$$G_i^p|_H=G^{p-1}, i_*:i^*H\rightarrow T(M).$$

Then, if the data is all $C^\infty$, there is a neighborhood $U$ of $N^{p-1}$ in $N^p$ and a unique parallel $C^\infty$ T.B.I. $G:T(U)\rightarrow T(M)$ that extends the initial data:

$$G|_H=G^{p-1}:H\rightarrow T(M) \text{ along } g|_U \text{ and }$$

$$G\cdot i_* = G^p:i^*T(N^p)\rightarrow T(M) \text{ along } g\cdot i.$$
\{Z_i, \ldots, Z_p = \partial/\partial z_p\} on \ V \ and \ \{Y_i, \ldots, Y_m\} along g|_V \ with \ the \ property \ that \ a \ C^\infty \ T.B.I. \ G \ defined \ along \ g|_V \ extends \ the \ initial \ data \ along \ g|_V \ if \ and \ only \ if \ its \ matrix \ representation \ (r_{ki}) \ with \ respect \ to \ these \ frames \ (G(Z_i) = \sum_k r_{ki}, Y_k, i = 1, \ldots, p) \ satisfies \ the \ equations

\[r_{ki} = \delta^{ki}, \quad k = 1, \ldots, m, \quad i = 1, \ldots, p - 1,\]

on \ V

and

\[r_{ip} = 0, \quad \ldots, \quad r_{p-1, p} = 0\]
on \ V \ \cap \ N^{p-1}.

It follows that the T.B.I.'s \ G \ that extend the initial data on \ V \ are in bijective correspondence with the \(m - p\) tuples \((r_{p+1, p}, \ldots, r_{mp})\) of \(C^\infty\) functions on \ V \ that vanish on \(N^{p-1} \cap V\). The condition that \ G \ be parallel along \ g|_V \ is expressed by the vanishing, for each \(n \in V\), of the projection of \(\sum_{i=1}^p \nabla z_i(n) G(Z_i) \) into \(LG(N^p)\). On some, perhaps smaller, neighborhood of \(\mathcal{H}\) this condition is equivalent to the Cauchy-Kowalewski system:

\[0 = \left< \sum_{i=1}^p \nabla z_i(n) G(Z_i), Y_j(n) \right> \]

\[= \left< \sum_{i=1}^{p-1} \nabla z_i(n) Y_i, Y_j(n) \right> + \sum_{l=p}^m r_{ip} \left( \nabla z_p(n) Y_l, Y_j(n) \right) + \frac{\partial r_{ip}}{\partial z_p}(n), \]

\[j = p + 1, \ldots, m.\]

Thus, on some sufficiently small neighborhood \(V^\#\) of any point \(\mathcal{H} \in N^{p-1}\), there is a unique \(C^\infty\) parallel T.B.I. \(G^\#\) that extends the initial data along \(g|_V^\#\). A neighborhood \(U\) of \(N^{p-1}\) in \(N^p\) can then be constructed on which there is a unique \(C^\infty\) parallel T.B.I. \(G\) that extends the initial data along \(g|_U\) so that for each \(\mathcal{H} \in N^{p-1}: G|_{UN^p} = G^\#|_{UN^p^\#}.

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