GROMOLL GROUPS, Diff $S^n$ AND BILINEAR CONSTRUCTIONS OF EXOTIC SPHERES

BY P. ANTONELLI, D. BURGHELEA AND P. J. KAHN

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1. Introduction and main results. The Kervaire-Milnor group $\Gamma^n$ has a filtration by subgroups,

$$0 = \Gamma^n_{a-1} \subset \cdots \subset \Gamma^n_{k+1} \subset \Gamma^n_k \subset \cdots \subset \Gamma^n_1 = \Gamma^n,$$

due to Gromoll [9], which we study by means of certain homomorphisms

$$\pi_p(SO_q) \otimes \pi_q(SO_p) \quad \Gamma^{p+1} \otimes \pi_q(SO_p) \quad \sigma_{p,q} \quad \tau_{p+1,q} \quad \Gamma^{p+q+1}$$

See [12] for definitions. The pairing $\sigma$ was first introduced by Milnor [13] and has been studied in [3], [11]. The pairing $\tau$ has been studied in [8], [10].

The groups of Gromoll are related to the homotopy groups of Diff $S^n$ by a simple pasting construction: namely, there are homomorphisms $\lambda_i: \pi_i(Diff S^n) \to \Gamma^{n+i+1}$ with image $\lambda_i = \Gamma^{n+i+1}_{i+1}$ (see Proposition 2.1 and also [9, §1]).

We shall detect nontrivial elements in some $\Gamma^{n}_{k+1}$. Note that $\Gamma^{n}_{k+1} \neq 0$ implies that $\Gamma^{n+i}_{i+1} \neq 0$ and, hence, $\pi_i(Diff S^{n+i+1-1}) \neq 0$, for all $i \leq k$. For slightly sharper statements see Proposition 3.3 and Proposition 3.4.

1.1. Theorem. (a) $\Gamma^{4k+1}_{2k-2} \neq 0$, for all $k \geq 4$.
(b) $\Gamma^{4k+2}_{2k} \neq 0$, for all $k \geq 0$, $k \neq 2l - 1$.

Here $v(k)$ is the maximum number of linearly independent vector fields on $S^{2k+1}$. It is well known that $v(k) = 1$ when $k$ is even and $v(k) \geq 3$, when $k$ is odd. Its precise value is given in [2].

Theorem 1.1 follows from some of our results on $\sigma$. Corollary 3.5, below, also based on work with $\sigma$, actually establishes fairly large lower bounds for the order of $\Gamma^{4k+1}_{2k-2}$ (with some restrictions on $k$).

AMS Subject Classifications. Primary 5710, 5755; Secondary 5322.

Key Words and Phrases. Kervaire-Milnor group of exotic spheres, $\Gamma^n$, Gromoll filtration of $\Gamma^n$, group of self-diffeomorphisms, homotopy type of CW complex, homotopy-abelian $H$-space, inertia groups of manifolds, sectional curvature, pinching,

bilinear pairings of Milnor-Munkres-Novikov.

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1.2. **Theorem.** (a) Let $Q$ be an odd prime, and let $u$ and $v$ be integers satisfying $0 < u < Q - 1$, $u - v 
eq Q - 1$. Write $n = 2(uQ + v + 1)(Q - 1) - 2(u - v) - 1$. Then, $\Gamma_{Q-2}^{n} \cong \mathbb{Z}_Q$. (b) $\Gamma_2^n$ and $\Gamma_0^n$ are nontrivial.

Theorem 1.2 is proved using $\tau$ (see Proposition 3.2). It generalizes results in [16].

1.3. **Theorem.** Diff $S^n$ cannot be dominated by a finite CW complex, provided $n \geq 7$.

In particular, for this range of values of $n$, Diff $S^n$ is not dominated by a finite-dimensional Lie group. This answers a question raised by J. Eells and R. Palais.

Theorem 1.3 contrasts with the fact that, for $n = 1, 2$, Diff $S^n$ has the homotopy type of $SO_{n+1}$ [18]. The only undecided dimensions, therefore, are $n = 3, 4, 5, 6$.

In §2 we deduce Theorem 1.3 from Theorem 1.1 (a) and Theorem 1.2 (b). In §3, we describe our results on $\sigma$ and $\tau$ and give a table of low-dimensional computations. In §4, we relate our results to the inertia groups $I(\Sigma^n \times S^n)$, and we comment on Gromoll's pinching numbers $\delta_n$.

We would like to thank J. Milnor and N. Kuiper for their stimulating suggestions.

2. **Proof of Theorem 1.3.** Diff $S^n$ (resp., Diff$(S^n, D^+_n)$) is the group of all $C^\infty$, orientation-preserving diffeomorphisms of $S^n$ (resp., those which keep fixed the upper hemisphere $D^+_n$). Give it the $C^\infty$ topology. $SO_{n+1}$ is a closed subgroup of Diff $S^n$. It is well known ([7], [17]) that Diff $S^n$ and Diff$(S^n, D^+_n)$ have the homotopy type of countable CW complexes and that the map $SO_{n+1} \times$ Diff$(S^n, D^+_n) \to$ Diff $S^n$ defined by group-multiplying the inclusions

$$SO_{n+1} \subset \text{Diff } S^n \supset \text{Diff } (S^n, D^+_n)$$

is a homotopy equivalence.

2.1. **Proposition.** (a) The multiplication of Diff$(S^n, D^+_n)$ is homotopy-abelian.

(b) Let $\lambda_i: \pi_i(\text{Diff } S^n) \to \Gamma_i^{n+1}$ be as in §1, and let $\mu_i$ be its restriction to the direct summand $\pi_i(\text{Diff } (S^n, D^+_n))$. Then image $\mu_i = \Gamma_i^{n+1}$.

Let $A_n = \text{Diff } (S^n, D^+_n)$ and note that $\pi_1 A_n = H_1 A_n$.

2.2. **Proof of Theorem 1.3.** If Diff $S^n \sim SO_{n+1} \times A_n$ is dominated by a finite CW complex, for some $n$, then so is $A_n$, and so $H_*(A_n; \mathbb{Z}_p)$ is finitely-generated, for all primes $p$. According to Browder [6],
therefore, $H_\bullet A_n$ has no torsion. In particular, $\pi_1 A_n = H_1 A_n$ is free-abelian. Thus, the projective class group $K_0(\pi_1 A_n)$ vanishes, so that $A_n$ has the homotopy type of a finite CW complex (Wall [21]). It now follows from Hubbuck [10] that the identity component of $A_n$ has the homotopy type of a point or of a product of circles, so that $\pi_i A_n = 0, i \geq 2$.

Theorem 1.1 (a) and Theorem 1.2 (b), together with Proposition 3.2 and the subsequent remark, imply that $\pi_1 A_7$ and $\pi_1 A_8$ have elements of finite order and that, for $n \geq 9$, there is some $i \geq 2$ such that $\pi_i A_n \neq 0$. Thus, $n \leq 6$ as desired. This completes our proof.

Note that when $\pi_1 A_n$ has elements of finite order Browder's theorem alone implies that $\text{Diff } S^n$ is not finitely dominated. Our results on the $\tau$-pairing (Theorem 1.2 and Proposition 3.2) yield infinitely many such $n$, but not enough to prove Theorem 1.3.

3. The pairings $\sigma$ and $\tau$. The Gromoll groups are related to $\sigma$ and $\tau$ by the next two propositions. Let $\mu_i:\pi_i(\text{Diff}(S^n, D^*_n)) \to \Gamma^{n+i+1}$ be as in Proposition 2.1 (b).

3.1. Proposition. For any $a, b$, $0 \leq a \leq q$, $0 \leq b \leq p$, let $i_a:\pi_p(SO_{q-a}) \to \pi_p(SO_q)$ and $i_b:\pi_q(SO_{p-b}) \to \pi_q(SO_p)$ be the homomorphisms induced by the standard inclusions. Write $c = a + b + 1$. Then, there is a homomorphism

$$
g_{c}:\pi_p(SO_{q-a}) \otimes \pi_q(SO_{p-b}) \to \pi_c(\text{Diff}(S^{p+a-q}, D^*_c))$$

such that $\mu_g = \sigma_{p,q} \circ (i_a \otimes i_b)$.

In particular, $\text{image } (\sigma_{p,q} \circ (i_a \otimes i_b)) \subset \Gamma^{p+q+1}$.

3.2. Proposition. For every $q > 1$, there is a homomorphism

$$h_q: \Gamma^{p+1} \otimes \pi_q(SO_p) \to \pi_q(\text{Diff}(S^p, D^*_p))$$

such that $\mu_{h_q} = \tau_{p+1,q}$.

In particular, $\text{image } \tau_{p+1,q} \subset \Gamma^{p+q+1}$.

Remark. Note that domain $\tau_{p+1,q}$ is finite, so that if image $\tau_{p+1,q}$ \neq 0, then $\pi_q(\text{Diff}(S^p, D^*_p))$ has elements of finite order.

To prove Theorem 1.2, we follow Novikov [16] and map $\tau_{p+1,q}$ into the composition pairing in stable homotopy. Then we apply results of Toda [19], [20].

The nonzero elements in Theorem 1.1 (b) are Kervaire spheres (which, of course, come from $\sigma$). We prove Theorem 1.1 (a), for large $k$, by applying the Eells-Kuiper $\mu$-invariant, as in [11], to Milnor’s plumbing construction [13] and by using the Barratt-Mahowald Splitting Theorem to show that $\mu$ takes the same values on image $\sigma_{4s-1,4s-1} \circ (i_a \otimes i_b)$ as on the entire image $\sigma_{4s-1,4s-1}$, provided $4s-1-a$
For small $k$, we use Milnor’s method [13] applied to the $\mu$-invariant.

For sharper results on $\sigma$, we generalize some work of D. R. Anderson [3] and again apply the Barratt-Mahowald Splitting Theorem. To describe our conclusions, let

$$j_m = \text{order image } J_{4m-1} \quad \text{and} \quad b_m = (2^{2m-1} - 1) B_m a_m j_m / 2m,$$

where $B_m$ is the $m$th Bernoulli number, and $a_m = 1$ or 2, according as $m$ is even or odd. Write

$$p_{r,s} = b_{r+s} / \gcd(b_{r+s}, b_r b_s).$$

3.3. PROPOSITION. Let $r$ and $s$ be integers satisfying $r\geq 6$, $s\geq 6$, $r < 2s < 4r$, and write $t = r + s$. Then, $\text{G}^{u+1}_{2t-2} \cap bP_{4t}$ contains a cyclic group of order $p_{r,s}$.

3.4. PROPOSITION. (a) Let $r$, $s$, $t$ be as in 3.3. Then $p_{r,s}$ is odd and

$$p_{r,s} > \frac{1}{8}(2t - 1) \left( \frac{2t - 2}{2r - 1} \right) j_t / j_s.$$

(b) Write $r = 2^d (2e+1)$. Then $p_{r,s} > 2^{2r-d-9}$.

REMARKS. The lower bound $\frac{1}{8}(2t-1) \left( \frac{2t-2}{2r-1} \right) j_t / j_s$ is often large. For example, if $r$ and $s$ are primes, $7 \leq r < s < 2r$. Then, this bound is larger than $2^{r+s-8} / (2r+1)(2s+1)$. Much stronger but more complicated statements are possible.

When $r = s$, Proposition 3.3 is essentially Anderson’s Theorem 1, [3], combined with Proposition 3.1. The proof of 3.4 involves complicated but elementary number theory.

We now display some divisors of $\text{G}_2^k$, $k$ and $n$ small. Results of [14], [15], [19], [20] are used for some of the calculations. Recall that $\text{G}_1^n = \text{G}_n$ and $\text{G}_{k+1}^n \subseteq \text{G}_k^n$. According to Cerf, $\text{G}_2^n = \text{G}_1^n$, for all $n$. For the reader’s convenience, we give the order of $\text{G}_2^n = \text{G}_1^n = \text{G}_n$ precisely.

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Some divisors of order (G^n)
When entries are omitted for \( n \leq 22 \), this means that our techniques give no additional information.

4. Remarks on \( I(\Sigma^{p+1} \times S^q) \) and the Gromoll numbers \( \delta_n \).

4.1. \( I(\Sigma \times S^q) \subset \Gamma_{q+1}^{p+1} \), for all \( \Sigma \in \Gamma^{p+1} \) and \( q \geq 2 \).
This follows from 3.2 and DeSapio's results on the \( \tau \)-pairing [8].

4.2. When \( p \geq 2q-1 \), some \( I(\Sigma^{p+1} \times S^q) \) have elements of odd prime order.
This follows from Theorem 2.1 and DeSapio [8], and it contrasts with the fact, deducible from [4], that \( I(\Sigma^{p+1} \times S^q) \) is 2-primary when \( p < 2q-1 \).

4.3. There are spheres in image \( \sigma \) which are not in image \( \tau \).
This follows from the last assertion in 4.2, together with 3.3 and 3.4 (a).

4.4. In [9], Gromoll defines an increasing sequence of real \( \delta_k \) satisfying \( \delta_1 = 1/4 \) and \( \lim \delta_k = 1 \). He proves that if the sphere \( \Sigma^n \) can be \( \delta_k \)-pinched, then \( \Sigma^n \subset \Gamma_k \). Since \( \Gamma_{n-2} = 0 \), [18], \( \Sigma^n \) can be \( \delta_{n-2} \)-pinched only if \( \Sigma^n \) is diffeomorphic to \( S^n \).

Question 1. Can every sphere in \( \Gamma_k \) be \( \delta_k \)-pinched?
This probably asks too much, since no examples of riemannian exotic spheres admitting positive sectional curvature are known.

Call \( \delta \) \( N \)-universal if \( 0 < \delta \leq 1 \) and if \( \Sigma^n \delta \)-pinched and \( n \geq N \) imply \( \Sigma^n \) diffeomorphic to \( S^n \).

Question 2. Does an \( N \)-universal \( \delta \) exist, for some \( N \)?
Question 2 was asked by Gromoll [9].

We simply remark here that an affirmative answer to either question implies a negative answer to the other, because \( \Gamma^{4k-2} \neq 0 \), \( k \geq 4 \).

References

1. J. F. Adams, On the groups \( J(X) \), IV, Topology 5 (1966), 21-71; correction, ibid., 7 (1968), 331. MR 33 #6628; MR 37 #5874.


Institute for Advanced Study, Princeton, New Jersey 08540