GENERALISED NUCLEAR MAPS IN NORMED LINEAR SPACES

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1. Preliminary definitions and notations. Grothendieck [3] and Pietsch [6] present an exhaustive study of nuclear operators and nuclear maps. The notion of a nuclear operator was extended by Persson and Pietsch in a recent paper [5] and they study in detail the \( p \)-nuclear and quasi-\( p \)-nuclear maps. In this paper we define and study certain linear maps called \( \lambda \)-nuclear and quasi-\( \lambda \)-nuclear maps. Our definition and generalisation here are motivated by the Köthe sequence spaces and their duality theory. For the special case \( \lambda = l^1 \) we obtain the nuclear operators and for \( \lambda = l^p \) we obtain the \( p \)-nuclear maps; also, the special case \( \lambda = c_0 \) yields the \( \infty \)-nuclear operators of Persson and Pietsch. Most of the results in this work are motivated by the work of Persson and Pietsch [5] and Köthe sequence spaces.

We shall briefly outline our assumptions. For definitions not stated here see Garling [1], Köthe [4], Ruckle [7], Sargent [9] and Zeller [10]. Let \( \lambda \) be a symmetric sequence space of scalars and \( \lambda^* \) be its Köthe dual. We shall assume that \( \lambda \) is provided with the Mackey topology of the duality \( (\lambda, \lambda^*) \) and that this topology is provided by a norm \( \rho \), \( \rho \) itself being an extended seminorm on \( \omega \). We assume now that \( \lambda \) is solid and that it is \( K \)-symmetric, i.e., for each \( x \in \lambda \) and for each permutation \( \pi \) of \( I^+ \) we have \( x_\pi \in \lambda \) and \( \rho(x_\pi) = \rho(x) \). \( \lambda \) is also assumed to be a BK space with AK. We remark that our assumptions imply that \( \lambda = \omega \) or \( \lambda = l^\infty \) or \( \lambda \subseteq c_0 \). The space \( \lambda^* \) is now considered as the topological dual of \( \lambda \) and equipped with its natural norm topology.

We pause now to point out that in addition to the spaces \( l^p \), \( 1 \leq p < \infty \), the sequence spaces \( n(\phi) \) of Sargent [8] and the sequence spaces \( \mu_{a,p} \) and \( v_{a,p} \) of Garling [2] serve as examples of the type of sequence spaces \( \lambda \) we consider. Garling shows also that his spaces \( \mu_{a,p} \) are in general not linearly homeomorphic to \( l^p \).

Next let \( E \) and \( F \) be normed linear spaces. Then \( \lambda(E) \) is the (vector sequence) space of all vectors \( x = (x_n), x_n \in E \) for each \( n \) and such that the sequence \( \langle x_n, a \rangle \in \lambda \) for each \( a \in E' \). Formally define

\[
\varepsilon_n(x) = \sup_{\|a\|_1 \leq 1} \rho(\| \langle x_n, a \rangle \|),
\]

where \( \rho \) is the norm on \( \lambda \).

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810
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$E$ is the space of sequences $x = (x_n)$, $x_n \in E$ for each $n$ and such that $(|x_n|) \in \lambda$; the space $E$ is equipped with a natural norm topology given by $||x|| = \rho(||x_n||)$.

2. $\lambda$-nuclear maps. Let $T$ be a linear map on the normed space $E$ into another, $F$. We define $T$ to be a $\lambda$-nuclear map if $T$ admits the representation

$$Tx = \sum_{n=1}^{\infty} \langle x, a_n \rangle y_n, \quad x \in E,$$

where $a = (a_n) \in \lambda [E']$ and $y = (y_n) \in \lambda^*(F)$ with $\rho(y) < \infty$. There may be other representations of $T$ in the above form. Keeping this in mind, we define

$$N_{\lambda}(T) = \inf \{ ||a|| \cdot \rho(y) \}$$

where the infimum is taken over all possible representations of $T$ in the above form.

We observe that $\lambda$-nuclear maps can be defined in the following equivalent way: say $T$ is $\lambda$-nuclear if $T$ has the representation

$$Tx = \sum_{n=1}^{\infty} \alpha_n \langle x, u_n \rangle y_n,$$

where $||u_n|| \leq 1$ for each $n$, $\alpha = (\alpha_n) \in \lambda$ and $y = (y_n) \in \lambda^*(F)$ with $\rho(y) \leq 1$. In this case

$$N_{\lambda}(T) = \inf \rho(\alpha).$$

Let $N_{\lambda}(E, F)$ denote the set of all $\lambda$-nuclear maps on $E$ into $F$.

**Theorem 1.** Each $\lambda$-nuclear map $T$ is continuous and $||T|| \leq N_{\lambda}(T)$.

**Theorem 2.** $N_{\lambda}(E, F)$ is a quasi-normed linear space under the norm $N_{\lambda}$; also if $F$ is a Banach space $N_{\lambda}(E, F)$ is complete if $\lambda$ is made of all sequences $u \in \omega$ for which $\rho(u) < \infty$.

**Theorem 3.** If $A(E, F)$ denotes the space of all operators $T$ on $E$ which have finite dimensional ranges in $F$, then $A(E, F)$ is a dense subspace of $N_{\lambda}(E, F)$.

**Corollary.** If $F$ is a Banach space then each $T \in N_{\lambda}(E, F)$ is a compact linear map and each such $T$ has a separable range space.

The next two theorems play an important role in the factorization theorem (Theorem 6) characterizing $\lambda$-nuclear maps.
THEOREM 4. Let $E$, $F$ and $G$ be normed linear spaces. Then we have the following:
(a) If $T \in N_{\lambda}(E, F)$ and $S \in L(F, G)$ then $S \circ T \in N_{\lambda}(E, G)$ and $N_{\lambda}(S \circ T) \leq \|S\| \cdot N_{\lambda}(T)$.
(b) If $T \in L(E, F)$ and $S \in N_{\lambda}(F, G)$ then $S \circ T \in N_{\lambda}(E, G)$ and $N_{\lambda}(S \circ T) \leq N_{\lambda}(S) \cdot \|T\|$. 

THEOREM 5. Let $\delta = (\delta_n)$ be a fixed member of $\lambda$. Then the map $D : l^{\infty} \rightarrow \lambda$ defined by $D(u) = (u, \delta_1)$ is a $\lambda$-nuclear map and $N_{\lambda}(D) = p(\delta)$.

THEOREM 6. Suppose $F$ is a Banach space. Then the map $T \in L(E, F)$ is $\lambda$-nuclear if and only if it can be factorized as follows:
$$T = Q \circ D \circ P, \quad E \rightarrow l^{\infty} \rightarrow \lambda \rightarrow F$$
where $P$ and $Q$ are continuous linear maps with $\|P\| \leq 1$ and $\|Q\| \leq 1$ and $D$ is as defined in Theorem 5.

3. Quasi-$\lambda$-nuclear maps. A linear map $T$ on $E$ into $F$ is defined to be quasi-$\lambda$-nuclear if there exists a sequence $a = (a_n)$ of elements of $E'$ such that $a \in \lambda[E']$ and $\|Tx\| \leq p(\langle x, a_n \rangle)$ for each $x \in E$. Set $Q_{\lambda}(T) = \inf \|a\|_T$, where the infimum is taken over all admissible $a$. Then one can prove that $Q_{\lambda}(E, F) \subseteq L(E, F)$ with $\|T\| \leq Q_{\lambda}(T)$. Also $N_{\lambda}(E, F) \subseteq Q_{\lambda}(E, F)$ with $Q_{\lambda}(T) \leq N_{\lambda}(T)$ for $T \in N_{\lambda}(E, F)$. In the opposite direction we have the following result.

THEOREM 7. If the Banach space $F$ has the extension property and if $T \in Q_{\lambda}(E, F)$ then $T \in N_{\lambda}(E, F)$ and $Q_{\lambda}(T) = N_{\lambda}(T)$.

We remark also that the above result is true for any pair $E$, $F$ provided the sequence space $\lambda$ is complemented. Thus for $\lambda = l^p$ when one gets the quasi-$2$-nuclear maps and the $2$-nuclear maps, we have the (known) result that $N_{\lambda}(E, F) = Q_{\lambda}(E, F)$.

4. $\lambda$-nuclear maps and absolutely $\lambda$-summing maps. The linear map $T$ on $E$ into $F$ is said to be absolutely $\lambda$-summing if for each $x = (x_n) \in \lambda(E)$, the sequence $Tx = (Tx_n) \in \lambda[F]$. Let now $\lambda = \{x \in \omega : \rho(x) < \infty\}$.

THEOREM 8. The linear map $T$ on $E$ into $F$ is absolutely $\lambda$-summing if and only if there exists a $\rho > 0$ such that for each finite system of vectors $x_1, x_2, \ldots, x_k$ in $E$ the following inequality holds:
$$\| (Tx_1, Tx_2, \ldots, Tx_k, 0, 0, \ldots) \|_T \leq \rho \cdot \delta_1(x_1, x_2, \ldots, x_k, 0, 0, \ldots).$$

The smallest such $\rho$ is denoted $\pi_\lambda(T)$. It can be shown that when $F$ is a Banach space the space $\pi_\lambda(E, F)$ of all the absolutely $\lambda$-sum-
ming maps on $E$ into $F$ is a Banach space with the norm defined by $\pi_\lambda(\cdot)$.

The space $\lambda$ is said to have the norm iteration property if for each sequence $(x^n)$ of elements of $\lambda$ we have $p[p(x^n)] = p[p(x_1)]$ where $x_1 = (x_1^1, x_1^2, \ldots, x_1^n, \ldots)$. It is easily verified that the spaces $c_0$ and $l^p (1 \leq p \leq \infty)$ have the above property.

**Theorem 9.** If $\lambda$ has the norm iteration property then $N_\lambda(E, F) \subseteq \pi_\lambda(E, F)$ and $\pi_\lambda(T) \subseteq N_\lambda(T)$.

We remark now that Theorem 9 above is true also for quasi-$\lambda$-nuclear maps with practically the same proof as that of Theorem 9. In case $\lambda = l^p (p \geq 1)$ the results of Persson and Pietsch [5] show that by taking the composition product of a certain finite number of $p$-quasi-nuclear maps one can obtain ultimately a nuclear map. In a rather general set up as ours we cannot prove a result of that type. Consequently when one attempts to formulate the concept of a $\lambda$-nuclear space using the standard canonical mappings, one obtains naturally two related concepts, those of $\lambda$-nuclear spaces and of quasi-$\lambda$-nuclear spaces.

**References**


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