RESEARCH PROBLEMS

The Research Problems department of the *Bulletin* has been discontinued. This final offering consists of recently rediscovered problems and solutions submitted before the closing of the department.

PROBLEMS.

1. Richard Bellman. *Orthogonal series*

Let $\{u_n(x)\}$ be an orthonormal sequence over the interval $[a, b]$ with the continuous, positive weight function $p(x)$. Let $f(x)$ be a continuous positive function in $[a, b]$ and let $f(x) \sim \sum_{n=1}^{N} a_n u_n(x)$. Does there exist a summability matrix (s_{nm}) such that $\sigma_N = \sum_{n=1}^{N} s_{nN} a_n u_n(x)$ converges to $f(x)$ as $N \to \infty$ and $\sigma_N(x) \geqq 0$ for $N \geqq 1$, $a \leqq x \leqq b$?

2. Richard Bellman. *Differential equations*

Let m_i, $i = 1, 2, \cdots, N$ be moments of a distribution, i.e., $m_i = \int_a^b x^i dG(x)$, $dG \geqq 0$. Consider the linear system $dx_i/dt = \sum_{j=1}^{N} a_{ij} x_j$, $x_i(0) = m_i$. What are necessary and sufficient conditions on the matrix $A = (a_{ij})$ so that the $x_i(t)$, $i = 1, 2, \cdots, N$, are moments for $t \geqq 0$?

3. Richard Bellman. *Approximation of functions*

Let $k(x, y)$ be a continuous function of x and y in the square $0 \leqq x, y \leqq 1$. Determine the minimum of $J(f, g) = \int_0^1 f(x)dx + \int_0^1 g(y)dy$, $k(x, y) \leqq f(x) + g(y)$. Consider the case where $k(x, y)$ is symmetric in x and y, and we ask that $f = g$.

Generalize both to the multidimensional case where

$$k(x_1, x_2, \cdots, x_N) \leqq f(x_1, x_2, \cdots, x_k) + g(x_{k+1}, x_{k+2}, \cdots, x_N),$$

and to the case where $J(f, g)$ is a more general functional.

Consider the case where $k(x_1, x_2, x_3)$ is symmetric in x_1, x_2, x_3, and we ask for the minimum of $\int_0^1\int f(x_1, x_2)dx_1dx_2$ subject to the condition that $k(x_1, x_2, x_3) \leqq f(x_1, x_2) + f(x_1, x_3) + f(x_3, x_1)$ and $f(x_1, x_2)$ is symmetric in x_1 and x_2.

Is there a systematic procedure for solving problems of this type involving the minimization of $J(g)$ where $f(p) \leqq g(p)$ and g is invariant under a group of operations?

4. George Brauer. *The L^p conjecture for a finitely additive measure*

Let $s = \{s_n\}$ be a sequence and let a point ρ_0 in I be fixed, where I is the unit interval $[0, 1)$ and the symbol X denotes the Stone-Cech compactification of the space X. Let

$$\int s d\mathcal{A} = \tilde{S}(\rho_0) \tag{1}$$

where $\tilde{S}$ denotes the extension of the function $S(r) = \sum_{n=0}^{\infty} s_n r^n$ to I, whenever this extension exists. Let $L^p(\mathcal{A})$ denote the space of sequences s such that $\int |s|^p d\mathcal{A} < \infty$. Let a multiplication of sequences be defined as follows

$$(s * t)_n = s_0 t_n + \sum_{k=1}^{n} (s_k - s_{k-1}) t_{n-k}.$$

For what values of p, $1 < \infty < \infty$, is $L^p(\mathcal{A})$ an algebra under this multiplication?

5. S. Charmonman. *Condition numbers of a test matrix*

The member of a set of nonsymmetric matrices that can be used in testing inversion algorithms is of the form A whose elements a_{ij} are given by

$$\begin{aligned} a_{ij} &= c_i, \qquad i \leqq j, \\ &= 1, \qquad i > j, \end{aligned}$$

where c_i can be arbitrarily chosen. The inverse of A obtained by inspection is B whose elements b_{ij} are given by

$$\begin{aligned} b_{ij} &= (-1)^{(j-i)}/(c_i - 1); && i = 1, 2, 3 \cdots (n-1); \\ & && j = i, i+1, \\ &= -1/[c_n(c_1 - 1)]; && i = n;\ j = 1, \\ &= [1/(c_{j-1} - 1) - 1/(c_j - 1)]/c_n; && i = n;\ j = 2, 3, 4 \cdots (n-1), \\ &= [1 + 1/(c_{n-1} - 1)]/c_n && i = j = n, \\ &= 0 && \text{otherwise.} \end{aligned}$$

What is the P-condition number (the ratio of the root of the largest absolute value of A to the root of the smallest absolute value of A) or the spectral condition number (the product of the spectral norm of A to that of B) in terms of c_i and the order of the matrix?

6. M. M. Chawla. *Approximation theory*

Let $f(x)$ be continuous for $0 \leqq x \leqq 1$. Define $\|f\| = \sqrt{(\int_0^1 f^2(x)dx)^{1/2}}$. Put $\phi(\lambda) = \|f - \lambda\|^2$. Then $\phi'(\lambda)$ is monotonically increasing and $\phi(\lambda)$ has a minimum at $\lambda = \lambda_0 = \int_0^1 f(x)dx$. Hence

$$\|f-\lambda\|^2 \geqq \|f-\lambda_0\|^2 = \int_0^1 f^2(x)dx - \left(\int_0^1 f(x)dx\right)^2.$$

Can any relations be established between the above deficit in the Cauchy-Schwarz inequality and $M = \max_{0\leqq x\leqq 1} f(x)$ and $m = \min_{0\leqq x\leqq 1} f(x)$? As it is invariant under the substitution $f(x) \to f(x) + h$, only $M-m$ could enter such a relation.

7. David E. Daykin. *A monotonicity conjecture*

Let W be a set of vectors $(i_1, i_2, \cdots, i_m)$, of various dimensions m, whose components i_ν are integers with $1 \leqq i_1 \leqq i_2 \leqq \cdots \leqq i_m$. Suppose that, if $(i_1, i_2, \cdots, i_m) \in W$ and $j_1, j_2, \cdots, j_n$ are positive integers such that $1 \leqq n \leqq m$ and $j_{\nu+1} - j_\nu \geqq i_{\nu+1} - i_\nu$ for $1 \leqq \nu < n$, then $(j_1, j_2, \cdots, j_n) \in W$. Assume further that $b_1, b_2, \cdots$, is a sequence of positive integers, and for every positive integer N there is one and only one vector $(i_1, i_2, \cdots, i_m)$ in W such that

$$N = b_{i_1} + b_{i_2} + \cdots + b_{i_m}.$$

Prove or disprove my conjecture that

$$(1) \qquad 1 = b_1 < b_2 < \cdots,$$

except in trivial cases, when $b_1, b_2, \cdots$, is a rearrangement of $s^0, s^1, s^2, \cdots$, for some positive integer s.

In [1] I proved the special case of this conjecture which shows that, when W is such that $(i_1, i_2, \cdots, i_m) \in W$ iff $i_{\nu+1} - i_\nu \geqq 2$ for $1 \leqq \nu < m$, then $b_1, b_2, \cdots$, is the Fibonacci sequence $1, 2, 3, 5, 8, \cdots$. Also I have unpublished proofs for certain special cases, some of which I described in [2]. The above method of representing integers includes, if we drop the uniqueness condition, several other methods which have appeared in the literature. Moreover in [2] I have described the relation between W and $b_1, b_2, \cdots$, under the assumption that (1) holds.

1. D. E. Daykin, *Representation of natural numbers as sums of generalised Fibonacci numbers*, J. London Math. Soc. **35** (1960), 143–161.

2. ———, *Representation of natural numbers as sums of generalised Fibonacci numbers*. II, Fibonacci Quart. (to appear).

8. Solomon W. Golomb. *Wilsonian products in groups*

The product $\prod_{g_i \in G} g_i$ extended over all distinct elements of the finite abelian group G is the identity element $1 \in G$, unless G contains a unique element of order 2, which we may call "-1", in which case

$\prod g_i = -1$. (A special case of this is Wilson's Theorem in elementary number theory.)

If G is finite nonabelian, consider the set P of products $\prod_{g_i \in G} g_i$ of all the distinct elements of G, taken in all possible permuted orders. If C denotes the commutator subgroup of G, then G/C is abelian, and P maps into a single element of G/C. Thus either $P \subset C$; or $P \subset -C$ if G/C has a unique element -1 of order 2. It is further clear that P is closed under all automorphisms and anti-automorphisms of G.

CONJECTURE. Prove (or disprove) that $P = \pm C$ for all finite groups. (The real difficulty lies in showing that P, or when applicable, $-P$, is closed under multiplication.)

9. Kenneth Loewen. *A generalization of Goldbach's conjecture*

For any even number A, can every number of the set $\{nA\}$ ($n = 1, 2, \cdots$), at least for n sufficiently large, be written as the sum of two primes, one of the form $pA - k$ and the other of the form $qA + k$ ($p = 1, 2, 3, \cdots; q = 0, 1, 2, \cdots$) where $0 \leqq k \leqq A/2$ and k and A are relatively prime?

For $A = 2$ and $k = 1$ this is Goldbach's conjecture.

There are counterexamples for small n for $A = 8$ and $k = 1$ and also $A = 10$; $k = 1$. However in both these cases it seems to work for larger n.

10. A. A. Mullin. *Algebraic structure of new residue systems*

It is a well-known result at the interface between number theory and group theory that a reduced residue system (mod n) is a (compatible) finite abelian group G [1]. Suppose one defines a *much reduced residue system* (mod n) as follows [2]: select only those residue classes of a reduced residue system (mod n) whose *least* positive elements satisfy the condition that their mosaics [3] have no prime in common with the mosaic of n. Such a much reduced system (mod n) is clearly a (compatible) finite abelian partial groupoid H [4] with identity, which is contained in G.

Note. G and H coincide when n is a prime number and, indeed, for infinitely many, but not all, composite numbers n.

Does there exist a natural number n such that H is either

(1) a *proper* submonoid [5] of G, or

(2) a *proper* subgroup of G?

If so,

(1)′ what are their structures,

(2)′ what positions do such H occupy in (a) the modular lattice of submonoids of G and (b) the modular lattice of subgroups of G, and

(3)′ can the foregoing technique be extended to nonabelian groups so as to be useful for producing candidates in a search for new simple groups of finite order?

1. W. J. LeVeque, *Topics in number theory*. Vol. 2, Addison-Wesley, Reading, Mass., 1956, pp. 6–7. MR **18**, 283.
2. A. A. Mullin, *On an analogue of the Ramanujan sum*, Notices Amer. Math. Soc. **13** (1966), 493–494. Abstract #66T-279.
3. ———, *Some related number-theoretic functions*, Bull. Amer. Math. Soc. **69** (1963), 446–447.
4. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*. Vol. 1, Math. Surveys, no. 7, Amer. Math. Soc., Providence, R. I., 1961, p. 1. MR **24** #A2627.
5. C. Chevalley, *Fundamental concepts of algebra*, Academic Press, New York, 1956, p. 3. MR **18**, 553.

11. E. Michael. *The topology of function spaces*

Let X be a compact metric space, and Y a CW-complex. Must the function space $C(X, Y)$, in the compact-open topology, be normal or even paracompact? (It is known that Y is always paracompact.)

The answer is unknown even in the special case where X is a closed interval, and Y consists of uncountably many closed intervals radiating from a point. The answer is affirmative if Y is countable (making $C(X, Y)$ Lindelöf) or locally finite (making $C(X, Y)$ metrizable).

12. M. G. Murdeshwar and S. A. Naimpally. *A conjecture on Banach spaces*

Prove or disprove: A Banach space is infinite dimensional if and only if it is isomorphic to a proper subset of itself.

13. E. D. Nix. *The topology of* E^ω

Does there exist a continuous surjective mapping of the real line E^1 onto the countably-infinite-dimensional Hilbert space E^ω $(=l_2)$? Equivalently, can E^ω be partitioned into a countable set of Peano spaces?

14. E. D. Nix. *Chaotic spaces*

By "Chaotic space" is meant a topological space of more than one point such that for every two distinct points p and q there exist neighborhoods, U of p and V of q, such that no nonempty open subset of U is homeomorphic to any open subset of V. In other words a "chaotic space" is a Hausdorff space of more than one point, no two of whose open sets are homeomorphic.

A. Do chaotic spaces exist?

B. Do chaotic spaces of the cardinality of the continuum exist?

C. Do completely normal, connected and locally connected chaotic spaces exist?

15. E. D. Nix. *Pairs of topologies*

Given any set S, of more than one element, and any two Hausdorff (T_2) topologies for S, does there necessarily exist a nonempty proper subset of S which is the interior of its own closure in both topologies? If not, what if the two topologies are further restricted to being completely normal?

16. R. M. Redheffer. *A class of weight functions*

Let $\{\lambda_n\}$ be a real sequence with $\lambda_n \neq 0$ and with signed counting-function $\Lambda_1(u)$; thus, $\Lambda_1(u)$ sgn u gives the number of λ_n between 0 and u. Similarly, $\{\mu_n\}$ is a real sequence with $\mu_n \neq 0$ and with signed counting-function $\Lambda_2(u)$. Setting $\Lambda = \Lambda_1 - \Lambda_2$, we assume that $\Lambda(u) = o(|u|)$ and form the meromorphic function

$$F(x) = \prod \frac{(1 - x/\lambda_n)e^{x/\lambda_n}}{(1 - x/\mu_n)e^{x/\mu_n}}.$$

For any even, positive integrable function $w(x)$ we denote by B_w the class of functions ϕ such that

$$\int_{-\infty}^{\infty} w(x) \,|\, \phi(x) \,|\, dx < \infty$$

and by B_w^+ the class admitting a majorant $\phi^+ \in B_w$ such that $\phi^+(x)$ sgn x increases. An elementary argument shows that if there exists a positive constant ρ such that $|x|^{1+\rho}w(x)$ sgn x decreases and $|x|^{3-\rho}w(x)$ sgn x increases, then

$$\Lambda(u) \in B_w^+ \quad \text{implies} \quad \log |F(x)| - x\int_{-|x|}^{|x|} \frac{\Lambda(u)}{u^2}\, du \in B_w.$$

What is the most general class of weights, w, having this property?

17. D. Suryanarayana. *Problems in theory of numbers*

(1) If $\alpha(x) = \prod_{p \le x}^{*} (1 - 1/p)$ where the asterisk in the product indicates that p runs through primes $\equiv 1 \pmod 4$, evaluate $\lim_{x\to\infty} \alpha^2(x) \log x$.

(2) If $\nu(n)$ is the Core (the maximal square-free divisor) of n and $\beta(x) = \sum_{n \le x} \nu(n)/n^2$, evaluate

$$\lim_{x\to\infty} \left\{\frac{\exp \beta(x)}{x^c}\right\}$$

for a suitable constant c.

(3) If $p^\alpha q_1^{2\beta_1} q_2^{2\beta_2} \cdots q_t^{2\beta_t}$, where $p \equiv \alpha \equiv 1 \pmod 4$, is an odd perfect number, then is

$$\sigma(q_1^{2\beta_1} q_2^{2\beta_2} \cdots q_t^{2\beta_t}) = p^\alpha$$

and

$$\sigma(p^\alpha) = 2q_1^{2\beta_1} q_2^{2\beta_2} \cdots q_t^{2\beta_t}?$$

18. Olga Taussky. *Problems on matrices and operators*

(1) Let A, B, C be three matrices with trace zero and elements in a field F. By a theorem of Shoda (Japan J. Math. **13** (1936), 361–368) and Albert and Muckenhoupt (Michigan Math. J. **4** (1957), 1–3) each of them can be represented as an additive commutator $ST-TS$. When can three matrices X, Y, Z be found such that $A=XY-YX$, $B=YZ-ZY$, $C=ZX-XZ$?

(2) It was shown by O. Taussky (von Mises Anniversary Volume, Academic Press, New York, 1954, pp. 67–68) that two nonsingular $n\times n$ matrices A, B can be expressed in the form $A=XYZ$, $B=ZYX$, with X, Y, Z again $n\times n$ matrices, if and only if $\det A=\det B$ (see also K. Fan, Arch. Math. **5** (1954), 102–107 and J. Brenner, Math. Rev. **16** (1955), 326). What happens in the case that A and B are singular? What is the operator analogue?

(3) Pairs of $n\times n$ matrices A, B with elements in a field F which can be expressed in the form $A=XY$, $B=YX$ with X, Y again $n\times n$ matrices in F have been characterized previously (Flanders, Proc. Amer. Math. Soc. **2** (1951), 871–874). What is the operator analogue?

(4) Let A and B be two operators. Find conditions for which the commutators $AB-BA$ or $A^{-1}B^{-1}AB$ (when it exists) can be replaced by commutators in which the factors are themselves commutators. In the case of finite matrices many results concerning this question are known.

Solutions.

1. Partial solution to Problem 9 of Vol. **68** (1962), p. 22, by Che-Bor-Lam.

In (1), the following problem is raised.

Prove or disprove: If $P(\alpha)$ is a polynomial of degree n, then $|P'(\alpha)/P(\alpha)|<1$ outside circles the sum of whose radius is less than An, where A is an absolute constant (p. 22, No. 9).

The aim of this note is to prove that the above statement is true if we replace An by $An\cdot\log n$. In this note, all Greek letters (with or without suffices) denote complex numbers, all small English letters denote integers and $r(B)$ denotes the radius of the circle B.

Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be the roots of $P(\alpha)$, and $A_0 = 3 \log n$.

Construct circles $B_1 B_2 \cdots B_r$ as follows:

Consider a circle of radius nA_0. If for a suitable centre, it can cover n roots, then let B_1 be the circle with radius nA_0 and that suitable centre. If the circle cannot cover n roots for any centre, we go on to consider circles of radius $(n-1)A_0$, and see whether it can cover $(n-1)$ roots. The above process is repeated until we can cover m roots with a circle of mA_0 and a suitable centre. Let that circle be B_1.

Now suppose $B_1 B_2 \cdots B_k$ $(r(B_k) = qA_0)$ are constructed and suppose $\alpha_1, \alpha_2, \cdots, \alpha_t$ are covered by $B_1 \cup B_2 \cup \cdots \cup B_k$, then consider a circle of radius qA_0. If it can cover q of the roots $R = \{\alpha_{t+1}, \alpha_{t+2}, \cdots, \alpha_n\}$ for a suitable centre, take the circle of radius qA_0 with that suitable centre to be B_{k+1}. If it cannot cover q of the roots in R, then go on to consider circles of radius $(q-1)A_0$ and see if it can cover $(q-1)$ roots in R. In this way, we can go on to construct B_{k+1}. This process is continued until all roots are covered with circles

$$\mathbf{B} = \{B_1, B_2, \cdots, B_s\}.$$

We have

$$\sum_{i=1}^{s} r(B_i) = nA_0.$$

We now construct a set of circles

$$\mathbf{C} = \{C_1, C_2, \cdots, C_s\}$$

where C_j is concentric with B_j and

$$r(C_j) = 2r(B_j) \qquad \text{for } j = 1, 2, \cdots, s.$$

Hence

$$\sum_{j=1}^{s} r(C_j) = 2nA_0 = 6n \log n.$$

LEMMA 1. *Let α be a point outside all circles of $\mathbf{C}$. Then a circle D_m with α as centre and radius mA_0 can cover at most $(m-1)$ roots.*

PROOF. Suppose D_m contains $\alpha_1, \alpha_2, \cdots, \alpha_m$. Then each α_i $(i = 1, \cdots, m)$ is covered by a certain circle of $\mathbf{B}$. Let B_r be the circle with the smallest suffix that contains one or more of $\alpha_1, \alpha_2, \cdots, \alpha_m$. If $r(B_r) = pA_0 \geqq mA_0$, then since $D_m \cap B_r \neq \varnothing$, therefore the distance between the centre of D_m and $B_r < 2pA_0$. Thus the centre of D_m is covered by C_r, contradicting the fact that α lies outside C_r. If $r(B_r) = pA_0 < mA_0$, then since B_r is the circle with the smallest suffix among all circles in $\boldsymbol{B}$ that contains one or more of $\alpha_1, \alpha_2, \cdots, \alpha_m$, then at the time when B_r is constructed, none of these roots are covered by any circle in $\boldsymbol{B}$. Hence a circle of radius mA_0 cannot cover the said m roots if $m > p$.

2. Solution to Problem 12 of Vol. **72** (1966), p. 470. The problem reads:

Consider the real differential equation

$$\dddot{x} + a\ddot{x} + \dot{x} + a \sin x = 0 \tag{1}$$

in which a is a constant. For sufficiently small x, the equation (1) takes the approximate form

$$\dddot{x} + a\ddot{x} + \dot{x} + ax = 0$$

which has periodic solutions of period 2π. It is thus reasonable to anticipate that the original equation (1) itself does have nontrivial periodic solutions for arbitrary values of a. Can this be proved? (Received December 31, 1965.)

Solution by N. Levinson (extract of a letter): ". . . there are no nontrivial period solutions of

$$\dddot{x} + a\ddot{x} + \dot{x} + a \sin x = 0, \qquad a \neq 0. \tag{1}$$

Indeed let ϕ be a solution of period T. Set $x = \phi$ in (1) and multiply by $\ddot{\phi} + a\dot{\phi}$. Integrate over a period to get

$$\int_0^T (1 - \cos \phi)\dot{\phi}^2 dt = 0 \cdots ."$$