RESEARCH PROBLEMS

The Research Problems department of the Bulletin has been discontinued. This final offering consists of recently rediscovered problems and solutions submitted before the closing of the department.

PROBLEMS.

1. Richard Bellman. Orthogonal series

Let \( \{u_n(x)\} \) be an orthonormal sequence over the interval \([a, b]\) with the continuous, positive weight function \( p(x) \). Let \( f(x) \) be a continuous positive function in \([a, b]\) and let \( f(x) \sim \sum_{n=1}^{N} a_n u_n(x) \). Does there exist a summability matrix \((s_{nm})\) such that

\[
\sum_{n=1}^{N} s_{nm} a_n u_n(x) \to f(x) \quad \text{as} \quad N \to \infty \quad \text{and} \quad \sigma_N(x) \geq 0 \quad \text{for} \quad N \geq 1, \quad a \leq x \leq b?
\]

2. Richard Bellman. Differential equations

Let \( m_i, i = 1, 2, \ldots, N \) be moments of a distribution, i.e.,

\[
m_i = \int_a^b x^i dG(x), \quad dG \geq 0.
\]

Consider the linear system

\[
dx_i/dt = \sum_{j=1}^{N} a_{ij} x_j,
\]

\( x_i(0) = m_i \). What are necessary and sufficient conditions on the matrix \( A = (a_{ij}) \) so that the \( x_i(t), i = 1, 2, \ldots, N \), are moments for \( t \geq 0? \)

3. Richard Bellman. Approximation of functions

Let \( k(x, y) \) be a continuous function of \( x \) and \( y \) in the square \( 0 \leq x, y \leq 1 \). Determine the minimum of

\[
J(f, g) = \int_0^1 f(x)dx + \int_0^1 g(y)dy + k(x, y) \leq f(x) + g(y).
\]

Consider the case where \( k(x, y) \) is symmetric in \( x \) and \( y \), and we ask that \( f = g \).

Generalize both to the multidimensional case where

\[
k(x_1, x_2, \ldots, x_N) \leq f(x_1, x_2, \ldots, x_N) + g(x_{k+1}, x_{k+2}, \ldots, x_N),
\]

and to the case where \( J(f, g) \) is a more general functional.

Consider the case where \( k(x_1, x_2, x_3) \) is symmetric in \( x_1, x_2, x_3 \), and we ask for the minimum of

\[
\int_0^1 \int_0^1 f(x_1, x_2)dx_1dx_2 \quad \text{subject to the condition}
\]

\( k(x_1, x_2, x_3) \leq f(x_1, x_3) + f(x_1, x_3) + f(x_3, x_1) \) and \( f(x_1, x_2) \) is symmetric in \( x_1 \) and \( x_2 \).

Is there a systematic procedure for solving problems of this type involving the minimization of \( J(g) \) where \( f(p) \leq g(p) \) and \( g \) is invariant under a group of operations?

4. George Brauer. The \( L^p \) conjecture for a finitely additive measure

Let \( s = \{s_n\} \) be a sequence and let a point \( p_0 \) in \( I \) be fixed, where \( I \) is the unit interval \([0, 1]\) and the symbol \( X \) denotes the Stone-Cech
compactification of the space \( X \). Let

\[
\int s \, d\alpha = S(\rho_0)
\]

where \( S \) denotes the extension of the function \( S(r) = \sum_{n=0}^{\infty} s_n r^n \) to \( I \), whenever this extension exists. Let \( L^p(\alpha) \) denote the space of sequences \( s \) such that \( \int |s|^p \, d\alpha < \infty \). Let a multiplication of sequences be defined as follows

\[
(s \ast t)_n = s_0 t_n + \sum_{k=1}^{n} (s_k - s_{k-1})t_{n-k}.
\]

For what values of \( p \), \( 1 < \infty < \infty \), is \( L^p(\alpha) \) an algebra under this multiplication?

5. S. Charmonman. *Condition numbers of a test matrix*

The member of a set of nonsymmetric matrices that can be used in testing inversion algorithms is of the form \( A \) whose elements \( a_{ij} \) are given by

\[
a_{ij} = c_i, \quad i \leq j,
\]

\[
= 1, \quad i > j,
\]

where \( c_i \) can be arbitrarily chosen. The inverse of \( A \) obtained by inspection is \( B \) whose elements \( b_{ij} \) are given by

\[
b_{ij} = (-1)^{i-j}/(c_i - 1);
\]

\[
i = 1, 2, 3 \cdots (n - 1); \quad j = i, i + 1,
\]

\[
= -1/[c_n(c_1 - 1)]; \quad i = n; \quad j = 1,
\]

\[
= [1/(c_{j-1} - 1) - 1/(c_j - 1)]/c_n; \quad i = n; \quad j = 2, 3, 4 \cdots (n - 1),
\]

\[
= [1 + 1/(c_{n-1} - 1)]/c_n; \quad i = j = n,
\]

\[
= 0 \quad \text{otherwise}.
\]

What is the \( P \)-condition number (the ratio of the root of the largest absolute value of \( A \) to the root of the smallest absolute value of \( A \)) or the spectral condition number (the product of the spectral norm of \( A \) to that of \( B \)) in terms of \( c_i \) and the order of the matrix?

6. M. M. Chawla. *Approximation theory*

Let \( f(x) \) be continuous for \( 0 \leq x \leq 1 \). Define \( \|f\| = \sqrt{\int_0^1 f^2(x) \, dx} \). Then \( \phi'(\lambda) \) is monotonically increasing and \( \phi(\lambda) \) has a minimum at \( \lambda = \lambda_0 = \int_0^1 f(x) \, dx \). Hence
$\left\| f - \lambda \right\|^2 \geq \left\| f - \lambda_0 \right\|^2 = \left( \int_0^1 f^2(x) dx - \left( \int_0^1 f(x) dx \right)^2 \right).$

Can any relations be established between the above deficit in the Cauchy-Schwarz inequality and $M = \max_{a \leq x \leq 1} f(x)$ and $m = \min_{a \leq x \leq 1} f(x)$? As it is invariant under the substitution $f(x) \rightarrow f(x) + h$, only $M - m$ could enter such a relation.

7. David E. Daykin. A monotonicity conjecture

Let $W$ be a set of vectors $(i_1, i_2, \ldots, i_m)$, of various dimensions $m$, whose components $i_v$ are integers with $1 \leq i_1 \leq i_2 \leq \cdots \leq i_m$. Suppose that, if $(i_1, i_2, \ldots, i_m) \in W$ and $j_1, j_2, \ldots, j_n$ are positive integers such that $1 \leq n \leq m$ and $j_{v+1} - j_v \geq i_{v+1} - i_v$ for $1 \leq v < n$, then $(j_1, j_2, \ldots, j_n) \in W$. Assume further that $b_1, b_2, \ldots$, is a sequence of positive integers, and for every positive integer $N$ there is one and only one vector $(i_1, i_2, \ldots, i_m)$ in $W$ such that

$$N = b_{i_1} + b_{i_2} + \cdots + b_{i_m}.$$ 

Prove or disprove my conjecture that

$$1 = b_1 < b_2 < \cdots,$$

except in trivial cases, when $b_1, b_2, \ldots$, is a rearrangement of $s^0, s^1, s^2, \ldots$, for some positive integer $s$.

In [1] I proved the special case of this conjecture which shows that, when $W$ is such that $(i_1, i_2, \ldots, i_m) \in W$ iff $i_{v+1} - i_v \geq 2$ for $1 \leq v < m$, then $b_1, b_2, \ldots$, is the Fibonacci sequence $1, 2, 3, 5, 8, \ldots$. Also I have unpublished proofs for certain special cases, some of which I described in [2]. The above method of representing integers includes, if we drop the uniqueness condition, several other methods which have appeared in the literature. Moreover in [2] I have described the relation between $W$ and $b_1, b_2, \ldots$, under the assumption that (1) holds.


8. Solomon W. Golomb. Wilsonian products in groups

The product $\prod_{a \in G} g_i$ extended over all distinct elements of the finite abelian group $G$ is the identity element $1 \in G$, unless $G$ contains a unique element of order 2, which we may call "$-1$", in which case
\[ \prod_{g_i} = -1. \] (A special case of this is Wilson's Theorem in elementary number theory.)

If \( G \) is finite nonabelian, consider the set \( P \) of products \( \prod_{g_i \in G} g_i \) of all the distinct elements of \( G \), taken in all possible permuted orders. If \( C \) denotes the commutator subgroup of \( G \), then \( G/C \) is abelian, and \( P \) maps into a single element of \( G/C \). Thus either \( P \subseteq C \); or \( P \subseteq C^{-1} \) if \( G/C \) has a unique element \(-1\) of order 2. It is further clear that \( P \) is closed under all automorphisms and anti-automorphisms of \( G \).

**Conjecture.** Prove (or disprove) that \( P = \pm C \) for all finite groups. (The real difficulty lies in showing that \( P \), or when applicable, \(-P\), is closed under multiplication.)

9. Kenneth Loewen. *A generalization of Goldbach's conjecture*

For any even number \( A \), can every number of the set \( \{nA\} \) \((n = 1, 2, \ldots)\), at least for \( n \) sufficiently large, be written as the sum of two primes, one of the form \( pA - k \) and the other of the form \( qA + k \) \((p = 1, 2, 3, \ldots; q = 0, 1, 2, \ldots)\) where \( 0 \leq k \leq A/2 \) and \( k \) and \( A \) are relatively prime?

For \( A = 2 \) and \( k = 1 \) this is Goldbach's conjecture.

There are counterexamples for small \( n \) for \( A = 8 \) and \( k = 1 \) and also \( A = 10; k = 1 \). However in both these cases it seems to work for larger \( n \).

10. A. A. Mullin. *Algebraic structure of new residue systems*

It is a well-known result at the interface between number theory and group theory that a reduced residue system (mod \( n \)) is a (compatible) finite abelian group \( G \) [1]. Suppose one defines a much reduced residue system (mod \( n \)) as follows [2]: select only those residue classes of a reduced residue system (mod \( n \)) whose least positive elements satisfy the condition that their mosaics [3] have no prime in common with the mosaic of \( n \). Such a much reduced system (mod \( n \)) is clearly a (compatible) finite abelian partial groupoid \( H \) [4] with identity, which is contained in \( G \).

*Note.* \( G \) and \( H \) coincide when \( n \) is a prime number and, indeed, for infinitely many, but not all, composite numbers \( n \).

Does there exist a natural number \( n \) such that \( H \) is either

1. a *proper* submonoid [5] of \( G \), or
2. a *proper* subgroup of \( G \)?

If so,

1'. what are their structures,
2'. what positions do such \( H \) occupy in (a) the modular lattice of submonoids of \( G \) and (b) the modular lattice of subgroups of \( G \), and
3'. can the foregoing technique be extended to nonabelian groups.
so as to be useful for producing candidates in a search for new simple groups of finite order?


11. E. Michael. *The topology of function spaces*

Let $X$ be a compact metric space, and $Y$ a CW-complex. Must the function space $C(X, Y)$, in the compact-open topology, be normal or even paracompact? (It is known that $Y$ is always paracompact.)

The answer is unknown even in the special case where $X$ is a closed interval, and $Y$ consists of uncountably many closed intervals radiating from a point. The answer is affirmative if $Y$ is countable (making $C(X, Y)$ Lindelöf) or locally finite (making $C(X, Y)$ metrizable).

12. M. G. Murdeshwar and S. A. Naimpally. *A conjecture on Banach spaces*

Prove or disprove: A Banach space is infinite dimensional if and only if it is isomorphic to a proper subset of itself.

13. E. D. Nix. *The topology of $E^a$*

Does there exist a continuous surjective mapping of the real line $E^1$ onto the countably-infinite-dimensional Hilbert space $E^a$ ($=l_2$)? Equivalently, can $E^a$ be partitioned into a countable set of Peano spaces?

14. E. D. Nix. *Chaotic spaces*

By “Chaotic space” is meant a topological space of more than one point such that for every two distinct points $p$ and $q$ there exist neighborhoods, $U$ of $p$ and $V$ of $q$, such that no nonempty open subset of $U$ is homeomorphic to any open subset of $V$. In other words a “chaotic space” is a Hausdorff space of more than one point, no two of whose open sets are homeomorphic.

A. Do chaotic spaces exist?

B. Do chaotic spaces of the cardinality of the continuum exist?
C. Do completely normal, connected and locally connected chaotic spaces exist?

15. E. D. Nix. Pairs of topologies

Given any set $S$, of more than one element, and any two Hausdorff ($T_2$) topologies for $S$, does there necessarily exist a nonempty proper subset of $S$ which is the interior of its own closure in both topologies? If not, what if the two topologies are further restricted to being completely normal?

16. R. M. Redheffer. A class of weight functions

Let $\{\lambda_n\}$ be a real sequence with $\lambda_n \neq 0$ and with signed counting-function $\Lambda_1(u)$; thus, $\Lambda_1(u) \operatorname{sgn} u$ gives the number of $\lambda_n$ between 0 and $u$. Similarly, $\{\mu_n\}$ is a real sequence with $\mu_n \neq 0$ and with signed counting-function $\Lambda_2(u)$. Setting $\Lambda = \Lambda_1 - \Lambda_2$, we assume that $\Lambda(u) = o(|u|)$ and form the meromorphic function
\[
F(x) = \prod \frac{(1 - x/\lambda_n)e^{x/\lambda_n}}{(1 - x/\mu_n)e^{x/\mu_n}}.
\]

For any even, positive integrable function $w(x)$ we denote by $B_w$ the class of functions $\phi$ such that
\[
\int_{-\infty}^{\infty} w(x) |\phi(x)| \, dx < \infty
\]
and by $B_w^+$ the class admitting a majorant $\phi^+ \subseteq B_w$ such that $\phi^+(x) \operatorname{sgn} x$ increases. An elementary argument shows that if there exists a positive constant $\rho$ such that $|x|^{1+\rho}w(x) \operatorname{sgn} x$ decreases and $|x|^{-\rho}w(x) \operatorname{sgn} x$ increases, then
\[
\Lambda(u) \subseteq B_w^+ \implies \log |F(x)| - x \int_{-|x|}^{|x|} \frac{\Lambda(u)}{u^2} \, du \in B_w.
\]

What is the most general class of weights, $w$, having this property?

17. D. Suryanarayana. Problems in theory of numbers

(1) If $\alpha(x) = \prod_{p \equiv 1 \pmod{4}}^*(1 - 1/p)$ where the asterisk in the product indicates that $p$ runs through primes $\equiv 1 \pmod{4}$, evaluate
\[
\lim_{x \to \infty} \alpha(x) \log x.
\]
(2) If $\nu(n)$ is the Core (the maximal square-free divisor) of $n$ and $\beta(x) = \sum_{n \leq x} \nu(n)/n^2$, evaluate
\[
\lim_{x \to \infty} \left\{ \frac{\exp \beta(x)}{x^e} \right\}
\]
for a suitable constant $c$.

(3) If $p^{a}q_1^{\alpha_1}q_2^{\alpha_2} \cdots q_i^{\alpha_i}$, where $p \equiv \alpha \equiv 1 \pmod{4}$, is an odd perfect number, then is

$$\sigma(q_1^{\alpha_1}q_2^{\alpha_2} \cdots q_i^{\alpha_i}) = p^a$$

and

$$\sigma(p^a) = 2q_1^{\alpha_1}q_2^{\alpha_2} \cdots q_i^{\alpha_i} ?$$

18. Olga Taussky. *Problems on matrices and operators*

(1) Let $A$, $B$, $C$ be three matrices with trace zero and elements in a field $F$. By a theorem of Shoda (Japan J. Math. 13 (1936), 361–368) and Albert and Muckenhoupt (Michigan Math. J. 4 (1957), 1–3) each of them can be represented as an additive commutator $ST - TS$. When can three matrices $X$, $Y$, $Z$ be found such that $A = XY - YX$, $B = YZ - ZY$, $C = ZX - XZ$?

(2) It was shown by O. Taussky (von Mises Anniversary Volume, Academic Press, New York, 1954, pp. 67–68) that two nonsingular $n \times n$ matrices $A$, $B$ can be expressed in the form $A = XYZ$, $B = ZYX$, with $X$, $Y$, $Z$ again $n \times n$ matrices, if and only if $\det A = \det B$ (see also K. Fan, Arch. Math. 5 (1954), 102–107 and J. Brenner, Math. Rev. 16 (1955), 326). What happens in the case that $A$ and $B$ are singular? What is the operator analogue?

(3) Pairs of $n \times n$ matrices $A$, $B$ with elements in a field $F$ which can be expressed in the form $A = XY$, $B = YX$ with $X$, $Y$ again $n \times n$ matrices in $F$ have been characterized previously (Flanders, Proc. Amer. Math. Soc. 2 (1951), 871–874). What is the operator analogue?

(4) Let $A$ and $B$ be two operators. Find conditions for which the commutators $AB - BA$ or $A^{-1}B^{-1}AB$ (when it exists) can be replaced by commutators in which the factors are themselves commutators. In the case of finite matrices many results concerning this question are known.

**Solutions.**


In (1), the following problem is raised.

Prove or disprove: If $P(\alpha)$ is a polynomial of degree $n$, then $|P'(\alpha)/P(\alpha)| < 1$ outside circles the sum of whose radius is less than $An$, where $A$ is an absolute constant (p. 22, No. 9).

The aim of this note is to prove that the above statement is true if we replace $An$ by $An \cdot \log n$. In this note, all Greek letters (with or
without suffices) denote complex numbers, all small English letters denote integers and \( r(B) \) denotes the radius of the circle \( B \).

Let \( \alpha_1, \alpha_2, \cdots, \alpha_n \) be the roots of \( P(\alpha) \), and \( A_0 = 3 \log n \).

Construct circles \( B_1 B_2 \cdots B_r \) as follows:

Consider a circle of radius \( nA_0 \). If for a suitable centre, it can cover \( n \) roots, then let \( B_1 \) be the circle with radius \( nA_0 \) and that suitable centre. If the circle cannot cover \( n \) roots for any centre, we go on to consider circles of radius \( (n-1)A_0 \), and see whether it can cover \( (n-1) \) roots. The above process is repeated until we can cover \( m \) roots with a circle of \( mA_0 \) and a suitable centre. Let that circle be \( B_r \).

Now suppose \( B_1 B_2 \cdots B_k \) \( (r(B_k) = qA_0) \) are constructed and suppose \( \alpha_1, \alpha_2, \cdots, \alpha_t \) are covered by \( B_1 \cup B_2 \cup \cdots \cup B_k \), then consider a circle of radius \( qA_0 \). If it can cover \( q \) of the roots \( R=\{\alpha_{t+1}, \alpha_{t+2}, \cdots, \alpha_m\} \) for a suitable centre, take the circle of radius \( qA_0 \) with that suitable centre to be \( B_{k+1} \). If it cannot cover \( q \) of the roots in \( R \), then go on to consider circles of radius \( (q-1)A_0 \) and see if it can cover \( (q-1) \) roots in \( R \). In this way, we can go on to construct \( B_{k+1} \). This process is continued until all roots are covered with circles

\[
B = \{B_1, B_2, \cdots, B_s\}.
\]

We have

\[
\sum_{i=1}^{s} r(B_i) = nA_0.
\]

We now construct a set of circles

\[
C = \{C_1, C_2, \cdots, C_s\}
\]

where \( C_j \) is concentric with \( B_j \) and

\[
r(C_j) = 2r(B_j) \quad \text{for } j = 1, 2, \cdots, s.
\]

Hence

\[
\sum_{j=1}^{s} r(C_j) = 2nA_0 = 6n \log n.
\]

**Lemma 1.** Let \( \alpha \) be a point outside all circles of \( C \). Then a circle \( D_m \) with \( \alpha \) as centre and radius \( mA_0 \) can cover at most \( (m-1) \) roots.

**Proof.** Suppose \( D_m \) contains \( \alpha_1, \alpha_2, \cdots, \alpha_m \). Then each \( \alpha_i \) \( (i=1, \cdots, m) \) is covered by a certain circle of \( B \). Let \( B_r \) be the circle with the smallest suffix that contains one or more of \( \alpha_1, \alpha_2, \cdots, \alpha_m \). If \( r(B_r) = pA_0 \geq mA_0 \), then since \( D_m \cap B_r \neq \emptyset \), there-
fore the distance between the centre of $D_m$ and $B_r < 2pA_0$. Thus the centre of $D_m$ is covered by $C_r$, contradicting the fact that $\alpha$ lies outside $C_r$. If $r(B_r) = pA_0 < mA_0$, then since $B_r$ is the circle with the smallest suffix among all circles in $B$ that contains one or more of $\alpha_1, \alpha_2, \ldots, \alpha_m$, then at the time when $B_r$ is constructed, none of these roots are covered by any circle in $B$. Hence a circle of radius $mA_0$ cannot cover the said $m$ roots if $m > p$.

2. Solution to Problem 12 of Vol. 72 (1966), p. 470. The problem reads:

Consider the real differential equation

\begin{equation}
\ddot{x} + a\dot{x} + \dot{x} + a \sin x = 0
\end{equation}

in which $a$ is a constant. For sufficiently small $x$, the equation (1) takes the approximate form

\begin{equation}
\ddot{x} + a\dot{x} + \dot{x} + ax = 0
\end{equation}

which has periodic solutions of period $2\pi$. It is thus reasonable to anticipate that the original equation (1) itself does have nontrivial periodic solutions for arbitrary values of $a$. Can this be proved?

(Received December 31, 1965.)

Solution by N. Levinson (extract of a letter): "... there are no nontrivial period solutions of

\begin{equation}
\ddot{x} + a\dot{x} + \dot{x} + a \sin x = 0, \quad a \neq 0.
\end{equation}

Indeed let $\phi$ be a solution of period $T$. Set $x = \phi$ in (1) and multiply by $\dot{\phi} + a\phi$. Integrate over a period to get

\[ \int_0^T (1 - \cos \phi)\dot{\phi}^2 dt = 0 \cdots. \]