NONLINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS 
AND THE GENERALIZED TOPOLOGICAL DEGREE

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Introduction. It is our purpose in the present note to present a 
general existence theorem for noncoercive elliptic boundary value 
problems for operators of the form:

\[ A(u) = \sum_{|\alpha| = m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u, \ldots, D^mu), \]

on closed subspaces \( V \) of the Sobolev space \( W^{m,p}(G) \), \( G \) an open subset 
of \( \mathbb{R}^n \), \( n \geq 1 \). This existence theorem is based upon an extension of the 
theory of the generalized topological degree for \( A \)-proper mappings 
of Banach spaces introduced in Browder-Petryshyn [8], [9], and, in 
particular, on an extension of the Borsuk-Ulam theorem to pseudo-
monotone mappings \( T \) from a reflexive separable Banach space \( V \) to 
its conjugate space \( V^* \).

To make a precise statement of our general existence theorem 
possible, we introduce the following notation: For a given \( m \geq 1 \), we 
let \( \xi \) be the \( m \)-jet of a function \( u \) from \( \mathbb{R}^n \) to \( \mathbb{R}^s \) for some given \( s \geq 1 \), 
i.e. \( \xi = \{ \xi_{\alpha}, |\alpha| = m \} \), and set

\[ \xi = \{ \xi_{\alpha} : |\alpha| = m \}, \quad \eta = \{ \eta_\beta : |\beta| \leq m - 1 \}, \]

where each \( \xi_\alpha, \xi_\beta, \) and \( \eta_\beta \) is an element of \( \mathbb{R}^s \). The set of all \( \xi \) of the 
above form is an Euclidean space \( \mathbb{R}^{rm} \), and correspondingly, \( \xi \in \mathbb{R}^{rm} \), 
\( \eta \in \mathbb{R}^{rm-1} \).

For each \( \alpha \), \( A_\alpha \) is assumed to be a function from \( G \times \mathbb{R}^m \) to \( \mathbb{R}^s \) satisfying 
the following conditions:

Assumptions on \( A(u) : (1) A_\alpha(x, \xi) \) is measurable in \( x \) for fixed \( \xi \) and 
continuous in \( \xi \) for fixed \( x \). For a given \( p \) with \( 1 < p < \infty \), there exists a 
constant \( c \) such that

\[ |A_\alpha(x, \xi)| \leq c \left( 1 + \sum_{|\beta| \leq m} |\xi_\beta|^{p_{\alpha\beta}} \right) \]

with \( p_{\alpha\beta} \leq (p-1) \) for \( |\alpha| = |\beta| = m \), and

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limit of \( A \)-proper mappings.
\[
\phi_{a, \beta} \leq \frac{np}{n - p(m - |\alpha|)} \leq \frac{n^p}{n - p(m - |\beta|)}, \quad \text{if } m - \frac{n}{p} \leq |\alpha| \leq m, \\
\quad m - \frac{n}{p} \leq |\beta| \leq m,
\]

\[
|\beta| + |\alpha| \leq 2m,
\]

\[
\phi_{a, \beta} \leq \frac{np}{n - p(m - |\alpha|)}, \quad \text{if } |\alpha| < m - \frac{n}{p},
\]

\[
m - \frac{n}{p} \leq |\beta| \leq m.
\]

(2) If \(\xi = (\xi', \eta)\), then for each \(x \in G, \eta \in \mathbb{R}^{n-1}\), \(\xi\) and \(\xi'\) in \(\mathbb{R}^n\) with \(\xi \neq \xi'\),

\[
\sum_{|a| = m} \langle A_a(x, \xi, \eta) - A_a(x, \xi', \eta), \xi_a - \xi'_a \rangle > 0,
\]

(\(\langle \cdot, \cdot \rangle\) denotes the inner product in \(\mathbb{R}^n\)).

(3) For each \(\gamma\) and \(\gamma'\) in \(\mathbb{R}^n\),

\[
\sum_{|a| = m} \langle A_a(x, \xi, \eta) - \gamma_a, \xi_a - \gamma'_a \rangle \to \infty \quad (|\xi| \to \infty),
\]

uniformly for bounded \(\eta\).

Let \(W^{m,p}(G)\) be the Sobolev space of \(s\)-vector functions \(u\) such that \(u\) and all its derivatives \(D^a u\) for \(|a| \leq m\) lie in \(L^p(G)\) where \(p\) is the exponent involved in the inequalities of Assumption (1). Then for any \(u\) and \(v\) in \(W^{m,p}(G)\), we may define the generalized Dirichlet form corresponding to the representation (1) by:

\[
(2) \quad a(u, v) = \sum_{|a| \leq m} (A_a(\xi(u)), D^a v),
\]

where

\[
\xi(u) = \{ D^a u : |a| \leq m \}, \quad A_a(\xi(u))(x) = A_a(x, \xi(u)(x)),
\]

\[
(w, v) = \int_G \langle w(x), u(x) \rangle dx, \quad \text{(integration with respect to Lebesgue } n\text{-measure)}.
\]

**Theorem 1.** Let \(G\) be an open subset of \(\mathbb{R}^n\) with \(G\) bounded and the Sobolev Imbedding Theorem valid on \(G\) (i.e. \(G\) satisfies mild smoothness conditions on its boundary). Let \(A(u)\) be a quasilinear elliptic operator of order \(2m\) on \(G\) of the form

\[
(1) \quad A(u) = \sum_{|a| \leq m} (-1)^{|a|} D^a A_a(\xi(u)),
\]
where the coefficient functions $A_\alpha$ satisfy Assumptions (1), (2), and (3) above. Suppose that $A(u)$ is odd in $u$, i.e. $A_\alpha(x, -\xi) = -A_\alpha(x, \xi)$ for each $\alpha$ and all $x$ in $G$, $\xi$ in $\mathbb{R}^m$. For each $w$ in $V^*$, the dual space of a closed subspace $V$ of $W^{m,p}(G)$, consider the problem of finding $u$ in $V$ such that $a(u, v) = (w, v)$ for all $v$ in $V$. Suppose that there exists a continuous function $\phi: \mathbb{R}^+ \to \mathbb{R}^+$ such that for each solution $u$ of this problem for any $w$ in $V^*$,

\[ \|u\|_V = \|u\|_{W^{m,p}(G)} \leq \phi(\|w\|_{V^*}). \]

Then for each $w$ in $V^*$, there exists at least one solution $u$ in $V$ of the problem: $a(u, v) = (w, v)$ for all $v$ in $V$.

We have used the notation $(w, v)$ in Theorem 1 to denote the pairing between $w$ in $V^*$ and $u$ in $V$.

**Theorem 2.** Let $G$ be a bounded, smoothly bounded open set in $\mathbb{R}^n$ (as in Theorem 1), and consider a one-parameter family of operators $A_t(u), t \in [0, 1]$, where for each $t$,

\[ A_t(u) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(u); t) \]

and the coefficient functions are continuous in $t$, uniformly for bounded $\xi$ and all $x$ outside a null set in $G$. For each $t$, we take the generalized Dirichlet form

\[ a_t(u, v) = \sum_{|\alpha| \leq m} (A_\alpha(u); t, D^n v), \]

where we assume that $A_t(u)$ satisfies Assumptions (1), (2), (3) for each $t$ in $[0, 1]$. Suppose that $A_1(u)$ is odd, and that there exists a continuous function $\phi: \mathbb{R}^+ \to \mathbb{R}^+$ such that if $a_t(u, v) = (w, v)$ for some $w$ in $V^*$, $u$ in $V$, $t$ in $[0, 1]$ and all $v$ in $V$, then $\|a_t(u, v) \leq \phi(\|w\|_{V^*})$.

Then the problem: $a_0(u, v) = (w, v)$ for all $v$ in $V$; has a solution $u$ in $V$ for each $w$ in $V^*$.

Theorem 2 includes Theorem 1 as the special case in which $A_t(u) = A(u)$ for all $t$ in $[0, 1]$. It also includes the standard existence theorem for $A(u)$ in which the Dirichlet form $a(u, v)$ is assumed to be coercive, i.e.

\[ \text{(6)} \quad \text{There exists } c: \mathbb{R}^+ \to \mathbb{R}^1 \text{ with } c(r) \to \infty \text{ as } r \to \infty \text{ such that } a(u, u) \geq c(\|u\|_V)\|u\|_V. \]

Indeed, if $A(u)$ is coercive, and if we set $A_t(u) = A(u) - ta(-u)$ for $t$ in $[0, 1]$, then $A_0(u) = A(u), A_1(u)$ is odd, the Assumptions (1), (2), and (3) hold for every $A_t(u)$, while since $a_t(u, u) = a(u, u) - ta(-u, u) = a(u, u) + ta(-u, -u)$, it follows that
provided that \( \|u\|_V > R \), where \( c(r) > 0 \) for \( r > R \). Suppose that for some \( u \) in \( V \), \( w \) in \( V^* \) and \( t \) in \([0, 1]\), we have

\[
a_t(u, w) = (w, v) \quad (v \in V).
\]

Then:

\[
c(\|u\|_V) \|w\|_V \leq a_t(u, u) = (w, u) \leq \|w\|_{V^*} \|u\|_V,
\]

and as a consequence \( c(\|u\|_V) \|w\|_V \leq \|w\|_{V^*} \) if \( u = 0 \). If we set \( \phi(s) = \sup \{ r : c(r) \leq s \} \), it follows that \( \|u\|_V \leq \phi(\|w\|_{V^*}) \) and by Theorem 2, the equation \( a(u, v) = (w, v) \) \( (v \in V) \), has a solution \( u \) in \( V \) for each \( w \) in \( V^* \).

Existence theorems for elliptic boundary problems of this type were first obtained by Višik [15] using compactness arguments and a priori estimates on \((m+1)st\) derivatives. Monotonicity arguments were first applied to these problems in Browder [2], [3], using the basic existence theorem for monotone maps from a reflexive Banach space \( V \) to \( V^* \) proved independently by Browder [2] and Minty [12]. The existence theorem in the coercive case was extended to elliptic operators \( A(u) \) satisfying Assumptions (1), (2), and (3) by Leray-Lions [11]. Borsuk-Ulam theorems for monotone and semimonotone operators in infinite dimensional Banach spaces were first derived by Browder [4], [5], and were first applied to odd, homogeneous, elliptic operators satisfying strong monotonicity conditions by Pohoţăev [14]. Theorem 1 was first obtained under a stronger hypothesis (3)' rather than (3) in Browder [6], together with Assumptions (1) and (2) on \( A(u) \). This is as follows:

(3)' There exist continuous functions \( k(\eta), k_0(\eta) > 0 \) such that

\[
\sum_{|\alpha| \leq m} \langle A_\alpha(x, \zeta, \eta) \xi_\alpha \rangle \geq k_0(\eta) \| \zeta \|^p - k(\eta),
\]

for all \( x \) in \( G \), \( \zeta \) in \( \mathbb{R}^m \), \( \eta \) in \( \mathbb{R}^{m-1} \).

1. Proofs of Theorems 1 and 2 rest upon general results concerning two classes of nonlinear mappings of monotone type from a reflexive Banach space \( V \) to its conjugate space \( V^* \).

DEFINITION 1. Let \( V \) be a Banach space, \( V^* \) its conjugate space, \( T \) a mapping from \( V \) to \( V^* \), Then:

(a) \( T \) is said to be pseudomonotone if for any sequence \( \{ u_j \} \) in \( V \) with \( u_j \) converging weakly to \( u \) in \( V \) such that \( \lim \sup (Tu_j, u_j - u) \leq 0 \), it follows that for any \( v \) in \( V \), \( \lim \inf (Tu_j, u_j - v) \geq (Tu, u - v) \).

(b) \( T \) is said to satisfy condition \((S)_+\) if for any sequence \( u_j \) in \( V \) with
\{u_j\} converging weakly to \( u \) in \( V \) for which \( \lim (Tu_j, u_j - u) \leq 0 \), it follows that \( u_j \) converges strongly to \( u \) in \( V \).

**Proposition 1.** Suppose that \( A \) satisfies Assumption (1). Then there exists a continuous bounded mapping \( T \) of \( V \) into \( V^* \) for a given closed subspace \( V \) of \( W^{m,p}(G) \) such that for all \( u \) and \( v \) of \( V \), \((Tu, v) = a(u, v)\). If \( A(u) \) satisfies Assumptions (2) and (3), \( T \) is pseudomonotone. If \( A(u) \) satisfies Assumptions (2) and (3)', then \( T \) satisfies condition \((S)_+\).

The proof of Proposition 1 is given in §1 of [7], and Appendix to §1. Pseudomonotonicity was first defined by Brézis in [1] (though our definition differs slightly from his in considering only sequences rather than filters). The condition \((S)_+\) was first defined in connection with the study of nonlinear eigenvalue problems in Browder [6] and is studied in detail in Browder [7], [8].

**Theorem 3.** Let \( V \) be a reflexive separable Banach space, \( T \) a mapping of \( V \) into \( V^* \) which is pseudomonotone, bounded on bounded sets, and continuous from each finite dimensional subspace of \( V \) to the weak topology of \( V^* \). Then:

(a) If \( T \) is an odd mapping outside of some ball around the origin and if \( T^{-1}(B) \) is bounded for any bounded subset \( B \) of \( V^* \), then \( R(T) \), the range of \( T \), is all of \( V^* \).

(b) If \( \{T_t\} \) is a family of bounded, pseudomonotone, finitely continuous mappings from \( V \) to \( V^* \) which is continuous in \( t \) uniformly on bounded subsets of \( V \), with \( T_0 = T \), \( T_t \) odd outside some ball, and if there exists a function \( \phi : \mathbb{R}^+ \to \mathbb{R} \) such that \( T_t(u) = w \) implies that

\[
\|w\| \leq \phi(t\|u\|) \quad (t \in [0, 1]),
\]

then \( R(T) = V^* \).

Theorem 3 and Proposition 1 together imply the validity of Theorems 1 and 2. Theorem 3 follows from an extension to the class of pseudomonotone mappings from \( V \) to \( V^* \) of the theory of the generalized degree defined for \( A \)-proper mappings of Banach spaces in Browder-Petryshyn [9], [10] and applied to mappings \( T \) from a reflexive \( V \) to \( V^* \) satisfying condition \((S)\) in Chapter 17 of Browder [8]. The basic facts are summarized in the following theorem:

**Theorem 4.** Let \( V \) be a reflexive separable Banach space, \( V^* \) its conjugate space. Let \( T \) be a mapping from \( V \) to \( V^* \) which is finitely continuous from \( V \) to \( V^* \) (i.e. continuous from each finite dimensional subspace of \( V \) to the weak topology of \( V^* \)) and bounded (i.e. maps bounded subsets of \( V \) into bounded subsets of \( V^* \)). Then:
(a) If $T$ is pseudomonotone, there exists a sequence \( \{ T_j \} \) of finitely continuous, bounded mappings, each satisfying condition \((S)_+\), which converges to $T$ uniformly on every bounded subset of $V$.

(b) If $T$ satisfies condition \((S)_+\), then $T$ is $A$-proper in the following sense \([9], [10]\): If $B$ is a closed ball of $V$, \( \{ V_j \} \) an increasing sequence of finite dimensional subspaces of $V$ whose union is dense in $V$, and if for each $j$, $u_j$ is an element of $V_j \setminus B$ such that for a given element $w$ of $V^*$,

$$\| \phi_j^* T u_j - \phi_j^* w \|_{V_j^*} \to 0 \quad (j \to \infty),$$

where $\phi_j$ is the injection map of $V_j$ into $V$, $\phi_j^*$ the projection map of $V^*$ onto $V_j^*$, then there exists an infinite subsequence \( \{ u_{j(k)} \} \) converging strongly to an element $u$ of $B$ such that $T(u) = w$.

The proof of Theorem 4 is given in Chapter 17 of Browder \([8]\). The second property tells us that the generalized degree theory of Browder-Petryshyn \([10]\) applies to mappings $T$ satisfying the condition \((S)_+\) (for the details of this application, see \([8]\)). The corresponding generalized degree theory for pseudomonotone maps follows from the convexity of the class of $T$ satisfying \((S)_+\) and the following theorem whose proof will be published elsewhere:

**Theorem 5.** Let $X$ and $Y$ be Banach spaces, $G$ a bounded open subset of $X$, and consider an oriented approximation scheme \( \{ (X_n, Y_n, P_n, Q_n) \} \) for mappings $T$ of $\text{cl}(G)$ into $Y$ in the sense of \([10]\). Let $Z$ be a convex family of $A$-proper mappings from $\text{cl}(G)$ to $Y$ with respect to the given approximation scheme. Let $T$ be a mapping from $\text{cl}(G)$ to $Y$ which is the uniform limit on $\text{cl}(G)$ of mappings $T_j$ from the class $Z$. Then:

(a) For any sequence \( \{ T_j \} \) from $Z$ converging to $T$, if $w$ does not lie in $\text{cl}(T(bdry(G)))$, then $\text{Deg}(T_j, G, w)$ is the same for all $j$ sufficiently large and does not depend upon the choice of $\{ T_j \}$. We denote this limit as $\text{Deg}(T, G, w)$.

(b) $\text{Deg}(T, G, w)$ is invariant under homotopy and weakly additive in the sense of Theorem 1 of \([10]\). If $\text{Deg}(T, G, w) \neq \{ 0 \}$ and if $T(\text{cl}(G))$ is closed in $Y$, then $w$ lies in $\text{cl}(T(G))$.

(c) If $T$ is odd in the sense of Theorem 1 of \([10]\), then $\text{Deg}(T, G, 0)$ consists only of odd integers, and $\text{Deg}(T, G, 0) \neq \{ 0 \}$.

**ADDED IN PROOF.** Results closely related to Theorem 5 have also been obtained by P. M. Fitzpatrick in connection with his Rutgers Ph.D. dissertation.
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