K-THEORETIC INTERPRETATION OF TAME SYMBOLS ON k(t)

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In [3] we introduced a canonical resolution for computing the Ktheory of [4] and we found a map $\psi: K_2(A) \rightarrow \kappa_2^{GL}(A)$ where $K_2(A)$ is the group defined by Milnor [5] and $\kappa_2^{GL}(A)$ is the group of [3]. The map ψ was proved surjective if A is a regular ring. In this announcement we indicated how to compute $\kappa_2^{GL}(k(t))$ for the field k(t) of rational functions in one variable t. As a biproduct of this work we have proved

THEOREM 1. Write $K_2(A[t, t^{-1}]) = K_2(A) \oplus X$. Then if A is regular, X has a homomorphic image $K_1(A)$.

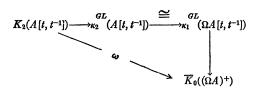
I should like to thank H. Bass for suggesting that Theorem 1, which was buried in my original announcement, be set off as a main result. Bass has informed me that J. Wagoner also has results on the group X.

1. Generalities. If R is any ring (without unit) recall the path ring $\Omega R = x(1-x)R[x]$. Clearly $\Omega(R[T]) = (\Omega R)[T]$ if T is a free abelian group or monoid. Also $\kappa_2^{GL}(R) \cong \kappa_1^{GL}(\Omega R)$ [3].

PROPOSITION 1¹. $\kappa_1^{GL}(R[t]) = \kappa_1^{GL}(R)$ and $\kappa_1^{GL}(R[t,t^{-1}]) = \kappa_1^{GL}(R)$ $\oplus \overline{K}_0(R^+)$.

This is an easy consequence of results of [1] and [3].

PROPOSITION 2. If A is regular, then the composition ω is a surjective homomorphism.



Theorem 1 follows from this proposition using results of [1] and [5].

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¹ ADDED IN PROOF. For the second conclusion of Proposition 1 we require that R be of the form $\Omega^n S$ where S is regular.

Let R[[t]] be the ring of formal power series in t and let $R((t)) = S^{-1}R[[t]]$ where $S = \{1, t, t^2, \cdots \}$.

PROPOSITION 3. There is a split epimorphism $K_1(R((t))) \rightarrow K_0(R)$.

The map is that defined on p. 74 of [1]. We prove that the module $M(\delta)$ defined there is finitely generated and projective over R. The proof of this fact is however necessarily different from that offered in [1].

COROLLARY. The epimorphism of Proposition 3 factors through the quotient $\kappa_1^{GL}(R((t)))$. Thus there is a commutative diagram

$$\begin{array}{ccc}
K_2(A((t))) \xrightarrow{\psi} \kappa_2^{GL}(A((t))) \to \kappa_1^{GL}(\Omega(A)((t))) \\
\downarrow \omega & \downarrow \\
\overline{K}_0((\Omega A)^+) \longleftarrow \kappa_1^{GL}(\Omega A)((t)))
\end{array}$$

PROPOSITION 4. If k is a field, then there is a canonical isomorphism $\overline{K}_0((\Omega k)^+) \cong k^*$.

This is established by applying the Mayer-Vietoris sequence [5] to the Cartesian square

$$(\Omega k)^+ \to (Ek)^+$$

$$\downarrow \qquad \downarrow \qquad (x \mapsto 1)^+$$

$$Z \longrightarrow k^+$$

2. Tame symbols. Let K be a field with a discrete valuation $v: K^* \rightarrow Z$, valuation ring A and residue class field L. Then the tame symbol [2] is a Steinberg symbol [3], [5]:

$$U(K) \times U(K) \xrightarrow{(,)_v} U(L) = L^*$$

defined as follows. Let $v(\pi) = 1$, and let $u, u_1 \in U(K)$. Write $u = a\pi^i$, $u_1 = b\pi^j$ where $a, b \in U(A)$. Then

$$(u, u_1)_v = \overline{(-1)^{ij}} \, \overline{a^{j/b^i}}$$

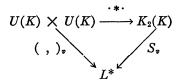
where "bar" is the residue class in L.

If A is a commutative ring, then there is a Steinberg symbol

$$U(A) \times U(A) \to K_2(A)$$
$$(u, u_1) \longmapsto u * u_1$$

defined by $u*u_1 = [h_{12}(u), h_{13}(u_1)]$. [5] and the symbol is natural with respect to ring homomorphisms. By the theorem of Matsumoto

[5], there is a unique homomorphism $S_{v}: K_{2}(K) \rightarrow L^{*}$ such that the following diagram commutes



In the case of k((t)) = K, A = k[[t]], L = k we have

PROPOSITION 5. If k is a field, the following diagram commutes up to sign

$$U(k((t))) \times U(k((t))) \xrightarrow{*} K_2(k((t)))$$
$$\downarrow (,)_v \qquad \qquad \downarrow \omega$$
$$k^* \longleftarrow \overline{K}_0(\Omega k^+).$$

COROLLARY. (,), factors through $\kappa_2^{GL}(k((t)))$.

Suppose now that p is an irreducible polynomial in k[t], determining the valuation ring A_p over k in k(t). Complete A_p in the padic topology to get the valuation ring \hat{A} , with residue class field L = k[t]/(p) and field of quotients k(t). By the structure theorem for complete local rings [6], $\hat{A} \cong L[[p]]$ and we have maps $k(t) \rightarrow k(t)$ $\cong L((p))$ yielding a commutative diagram

This diagram, together with the corollary to Proposition 5 applied to L((p)) implies

PROPOSITION 6. The tame symbol $(,)_p$ on k(t) determined by the irreducible polynomial $p \in k[t]$ factors through $\kappa_2^{GL}(k((t)))$. That is, there is a commutative diagram

$$U(k(t)) \times U(k(t)) \to K_2(k(t))$$

$$\downarrow (,)_p \qquad \downarrow \psi$$

$$L^* \to \kappa_2^{GL}(k(t))$$

where L = k[t]/(p).

As a consequence of Proposition 6 and results of [2] we have

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THEOREM 2. If k is a field, there is a split exact sequence

$$0 \to \kappa_2^{GL}(k) \to \kappa_2^{GL}(k(t)) \to \oplus U(k[t]/(p)) \to 0$$

where the sum is taken over all monic irreducible polynomials p in k[t].

I suspect that an analogue of Proposition 6 is valid for any global field. However, one of the main tools in this work, the results of [1], is not available for discrete valuation rings in the unequal characteristic case.

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