EXTENSION THEORY FOR CONNECTED HOPF ALGEBRAS

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1. Introduction. Let $K$ be a fixed commutative ring with unit. We will deal with graded algebras, coalgebras, and Hopf algebras over $K$ as defined in Milnor-Moore [4], but we assume that the underlying $K$-modules are connected.

Suppose $A$, $B$ are Hopf algebras; $A$ commutative and $B$ cocommutative. By an extension of $B$ by $A$ we mean a diagram of Hopf algebras and Hopf maps

$$
\begin{array}{ccc}
A & \rightarrow & C \\
\alpha & \mapsto & \beta \\
\downarrow & & \downarrow \\
B & \rightarrow & B
\end{array}
$$

(1.1)

in which $C$ is isomorphic to $A \otimes B$ simultaneously as a left $A$-module and right $B$-comodule. In this paper we announce results which describe and classify all extensions by $B$ by $A$. Proofs will appear in [5].

2. Matched pairs. If $B$ is an algebra we write

$$
\eta: K \rightarrow B, \quad \mu_B: B \otimes B \rightarrow B
$$

for the unit and multiplication, respectively. If $A$ is a coalgebra we write

$$
\varepsilon: A \rightarrow K, \quad \psi_A: A \rightarrow A \otimes A
$$

for the counit and comultiplication.

As the first step in classifying extensions, we will show in [5] how a diagram (1.1) determines a pair of $K$-linear maps

$$
\sigma_A: B \otimes A \rightarrow A, \quad \rho_B: B \rightarrow A.
$$

$\sigma_A$ is the “action” of base on fiber that one expects in an extension problem; $\rho_B$ is its dual. We prove:

(a) $\sigma_A$ gives $A$ the structure of a left $B$-module algebra;
(b) $\rho_B$ gives $B$ the structure of a right $A$-comodule coalgebra;
(c) the diagram commutes:

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Conversely, suppose given a pair of Hopf algebras \((A, B)\) with \(A\) commutative and \(B\) cocommutative, and suppose given \(K\)-linear maps \(\sigma_A, \rho_B\) satisfying conditions (a)-(d). Then we call \((A, B)\) a “matched pair”.

3. Bimodules over a matched pair. Suppose \((A, B)\) is a matched pair. Suppose \(N\) is simultaneously a left \(B\)-module under \(\sigma_N: B \otimes N \rightarrow N\), and a right \(A\)-comodule under

\[\rho_N: N \rightarrow N \otimes A.\]

Then we call \(N\) an \((A, B)\)-bimodule if the diagram commutes:
For example, if \((A, B)\) is a matched pair we can give \(A\) the structure of an \((A, B)\)-bimodule, with left \(B\)-action \(\sigma_A : B \otimes A \rightarrow A\), and right \(A\)-coaction \(\psi_A : A \rightarrow A \otimes A\). A dual construction makes \(B\) into an \((A, B)\)-bimodule. We say \(f : M \rightarrow N\) is a map of \((A, B)\)-bimodules if \(f\) is simultaneously a map of left \(B\)-modules and right \(A\)-comodules.

The interpretation of diagrams (2.1), (2.2), (3.1) is found in:

**Theorem 3.1.** Let \(N\) be a bimodule over the matched pair \((A, B)\). Let the map \(\tilde{\sigma}_N \otimes \check{\sigma}_A : B \otimes N \otimes A \rightarrow N \otimes A\) be the composition

\[
(N \otimes \mu_A)(\sigma_N \otimes A \otimes \sigma_A)(1, 4, 2, 3, 5)(\rho_B \otimes B \otimes N \otimes A)(\psi_B \otimes N \otimes A).
\]

Let the map \(\tilde{\rho}_N \otimes \check{\rho}_A : N \otimes A \rightarrow N \otimes A \otimes A\) be \(N \otimes \psi_A\). Then:

(a) \(\tilde{\sigma}_N \otimes \check{\sigma}_A\) gives \(N \otimes A\) the structure of a left \(B\)-module;
(b) \(\tilde{\rho}_N \otimes \check{\rho}_A\) gives \(N \otimes A\) the structure of a right \(A\)-comodule;
(c) with these structure maps \(N \otimes A\) is in fact an \((A, B)\) bimodule which we denote \(N \otimes A\);
(d) \(\rho_N : N \rightarrow N \otimes A\) is a map of \((A, B)\)-bimodules.

Theorem 3.1 has a dual. Given an \((A, B)\)-bimodule \(M\), Theorem (3.1)* tells how to give \(B \otimes M\) the structure of an \((A, B)\)-bimodule, denoted \(B \otimes M\), in such a way that \(\sigma_M : B \otimes M \rightarrow M\) is a map of \((A, B)\)-bimodules.

4. \((A, B)\)-algebras and \((A, B)\)-coalgebras. By an \((A, B)\)-algebra we mean an \((A, B)\)-bimodule \(N\) that is also a commutative algebra, in such a way that \(\mu_N : N \otimes N \rightarrow N\) is both a map of left \(B\)-modules and right \(A\)-comodules. \((A, B)\)-coalgebras are defined similarly. For example if \((A, B)\) is a matched pair, then \(A\) itself is an \((A, B)\)-algebra, and \(B\) is an \((A, B)\)-coalgebra.

**Theorem 4.1.** Let \(N\) be an \((A, B)\)-algebra. Let \(N \otimes A\) have the algebra structure of the tensor product \(N \otimes A\). Then \(N \otimes A\) is an \((A, B)\)-algebra, and \(\rho_N : N \rightarrow N \otimes A\) is a map of \((A, B)\)-algebras.
Theorem 4.1 can be interpreted in the language of “triples” [1], [2]. Let $\Gamma$ be the category of $(A, B)$-algebras. Let $S: \Gamma \to \Gamma$ be the functor which carries $N$ to $N \tilde{\otimes} A$, and let $I: \Gamma \to \Gamma$ be the identity functor. Define functor transforms $\delta: I \to S$, $\sigma: S^2 \to S$ by

$$\delta(N) = \rho_N: N \to N \tilde{\otimes} A; \quad \sigma(N) = N \otimes \epsilon \otimes A: N \tilde{\otimes} A \tilde{\otimes} A \to N \tilde{\otimes} A.$$

Then $V \equiv (S, \delta, \sigma)$ is a triple on the category $\Gamma$.

Similarly, the dual of Theorem 4.1 gives a cotriple $W \equiv (T, d, s)$ on the category $\Delta$ of $(A, B)$-coalgebras. Here $T(M) = B \otimes M$, $d(M) = \sigma_M: B \otimes M \to M$, and $S(M) = B \otimes \eta \otimes M: B \otimes M \to B \otimes B \otimes M$.

5. The cohomology of matched pairs. If $\phi$ is any category, write $S^*(\phi)$ for the category of cosimplicial objects over $\phi$, and $S_*(\phi)$ for the category of simplicial objects over $\phi$. Then the triple $V$ of §4 gives rise in the usual way [1], [2] to a functor $V: \Gamma \to S^*\Gamma$; the cotriple $W$ gives rise to a functor $W: \Delta \to S\Delta$. For example, if $(A, B)$ is a matched pair, then $V(K)$ is the acyclic cobar construction on $A$, but it has some structure not present in the classical case . . . an action of $B$ compatible with the coface operators. Similarly, $W(K)$ is the acyclic bar construction on $B$, with an $A$-coaction added.

If $M$ is an $(A, B)$-coalgebra and $N$ an $(A, B)$-algebra, let $\text{Hom}_{(A, B)}(M, N)$ denote the set of maps of $(A, B)$-modules $f: M \to N$ for which $f_0: M_0 \to N_0$ is the identity on $K$. Then $\text{Hom}_{(A, B)}(M, N)$ is an abelian group under the composition law $f + g = \mu_N(f \otimes g)\eta_M$.

Now to any matched pair $(A, B)$ we associate a double cosimplicial abelian group $X(B, A)$ by setting:

$$X^{p, q}(B, A) = \text{Hom}_{(A, B)}(W(K)_p, V(K)_q).$$

Let $\overline{X}(B, A)$ be the associated “total” cochain complex. Then we define the cohomology of the matched pair $(A, B)$ by:

$$H^*(B, A) = H^*(\overline{X}(B, A)).$$

6. Classification of extensions. Let $(A, B)$ be a matched pair under $(\sigma_A, \rho_B)$. Denote by $\text{Opext}(B, A)$ the set of equivalence classes of extensions (1.1) which give rise to the given “matching.” Our main result is:

**Theorem 6.1.** There is a natural isomorphism:

$$H^0(B, A) = \text{Opext} (B, A).$$

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REFERENCES


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